ACTUARIAL SOCIETY OF SOUTH AFRICA

THE QUANTIFICATION OF TYPE-2 PRUDENCE IN ASSET ALLOCATION BY THE TRUSTEES OF A RETIREMENT FUND

Presented by: Rob Thomson

Co-author: Taryn Reddy

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The quantification of type-2 prudence

1. Introduction
2. General review of literature
3. Definition of the utility function
4. Decreasing relative risk aversion
5. Counter-intuitive results and the challenge to prudence
6. Summary

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3. Definition of the utility function
4. Decreasing relative risk aversion
5. Counter-intuitive results and the challenge to prudence
6. Summary

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5. Counter-intuitive results and the challenge to prudence
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2. General review of literature
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6. Summary

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ACTUARIAL SOCIETY OF SOUTH AFRICA

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3. Definition of the utility function
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3. Definition of the utility function
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Introduction

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   - they have fiduciary responsibilities towards beneficiaries; and

   - they are required to be prudent.

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General review of the literature

- **The requirement of prudence**
- Behavioural finance
- The normative validity of EU theory for decision-making by a trustee
- Other measures of risk
- The application of EU theory to asset allocation

General review of the literature

- The requirement of prudence
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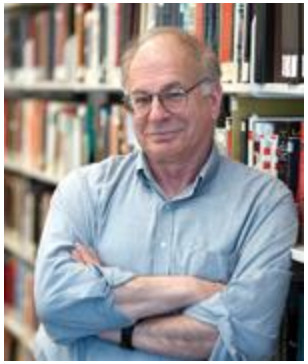
General review of the literature

Tversky & Kahneman (1986):

"The … normative and the descriptive analyses of choice should be viewed as separate enterprises."



Daniel Kahneman

Shiller (2003):

"… both approaches to finance, the behavioral approach, and the rational optimizing approach, have their own contributions to make …"

Bob Shiller

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- **The argument of the utility function: DC funds**
- **The argument of the utility function: DB funds**
- Dynamic Asset Allocation
- **The functional form of the utility function**
- Separation of value and risk and the use of discontinuities
- Group decision-making
- **Levels of risk aversion and prudence**

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The argument of the utility function: the DC benefit ratio

$$z_T = \frac{A_T \cdot {}_{T-t}a_0}{A_0 . a_T}$$

where:

A_T is the member's accumulated balance at time T, being her/his retirement date;

a_T is the price per unit, at time T, of an inflation-protected immediate annuity;

A_0 is the member's current balance (at time 0); and

${}_{T-t}a_0$ is the price per unit of an inflation-protected deferred annuity payable with effect from retirement, notionally purchased at time 0.

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The argument of the utility function: the DB benefit ratio

$$z_t = \frac{A_t + P_t}{\dot{L}_t + \dot{P}_t}$$

where:

A_t is the value of the assets of the fund available at time t for subsequent benefits;

$\dot{L}_t$ is the value of the liabilities of the fund at time t for subsequent reasonable expectations of benefits in respect of service to time 0;

P_t is the value of payments actually made during the period $[0, t]$, accumulated to time t with interest at the risk-free rate from time to time; and

$\dot{P}_t$ is the value of payments of reasonable benefit expectations during the period $[0, t]$, accumulated to time t with interest at the risk-free rate from time to time.

The argument of the utility function: DB funds

$$z_t = \frac{A_t + P_t}{\dot{L}_t + \dot{P}_t}$$

$$z_0 = \frac{A_0}{\dot{L}_0}$$

$$z_T = \frac{A_T + P_T}{\dot{P}_T}$$

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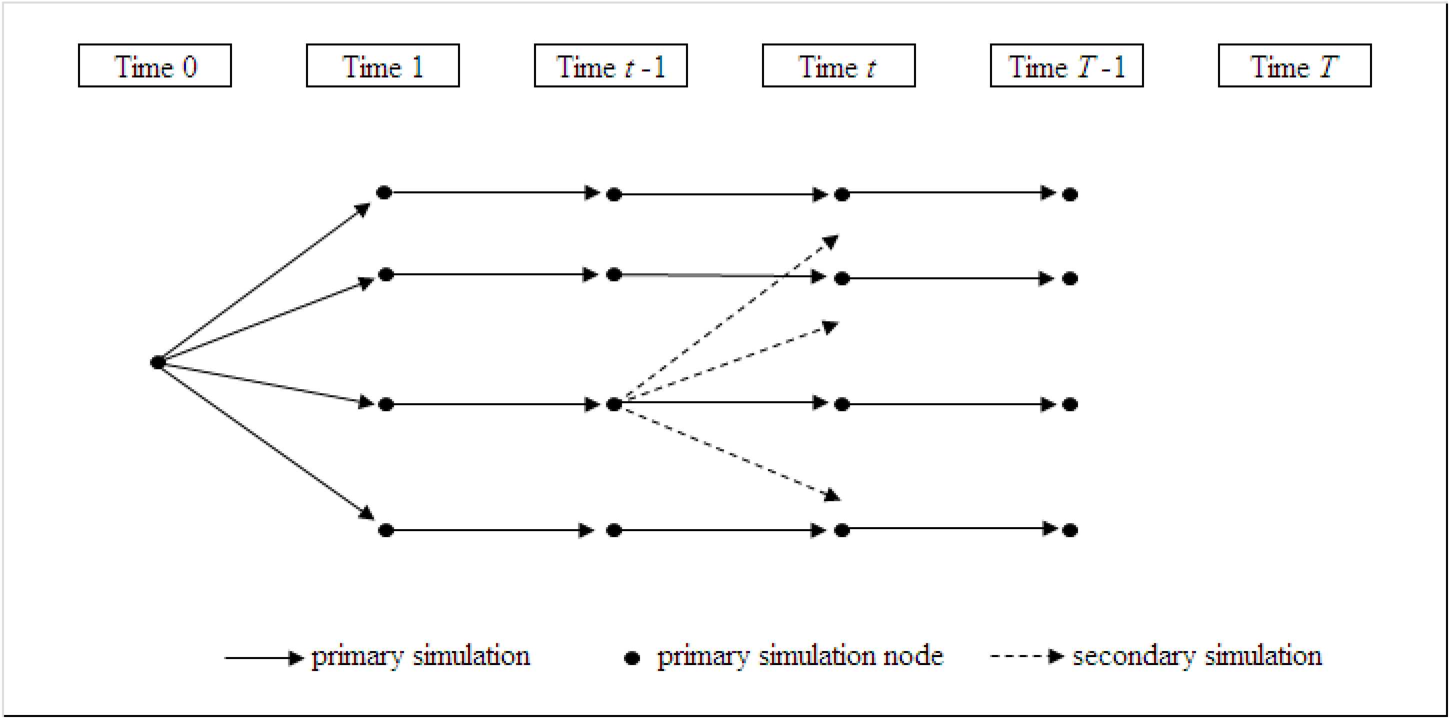
Dynamic asset allocation



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The functional form of the utility function

- Constant absolute risk aversion:

$$\lambda(z) = -\frac{u''(z)}{u'(z)} = \lambda \Rightarrow u(z) = -\exp\{-\lambda z\}$$

- Constant relative risk aversion:

$$\gamma(z) = -z\frac{u''(z)}{u'(z)} = \lambda \Rightarrow u(z) = \frac{z^{1-\gamma}-1}{1-\gamma} \text{ for } \gamma > 1$$

The functional form of the utility function

- Constant absolute risk aversion:

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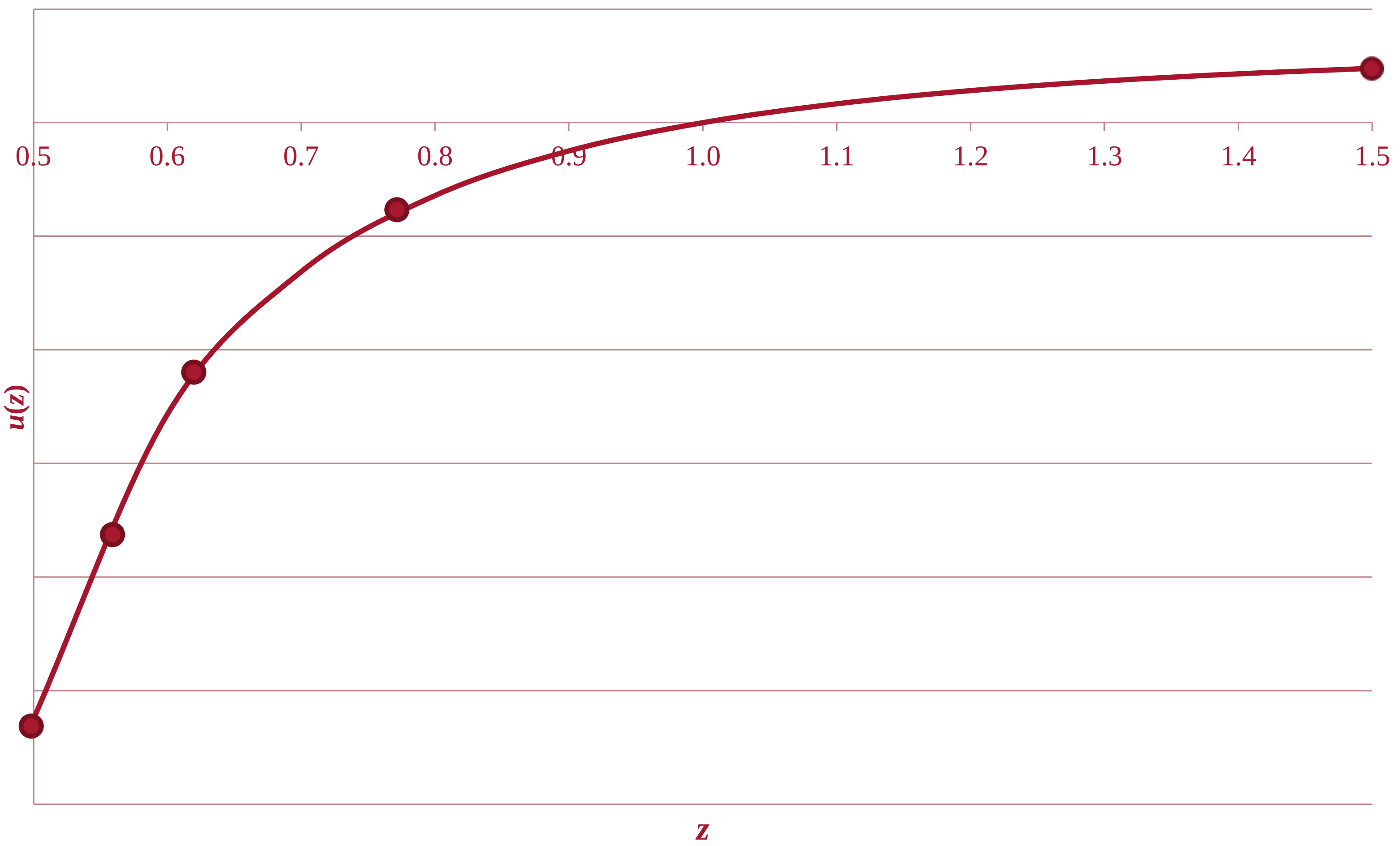
- HARA utility function:

$$\lambda(z) = -\frac{u''(z)}{u'(z)} = \frac{1}{a+bz} \Rightarrow u(z) = \begin{cases} \frac{\gamma}{1-\gamma}\left(\frac{\alpha z}{\gamma} + \beta\right)^{1-\gamma} \text{ for b} \neq 0, \gamma \neq 1; \\ \ln(z+\beta) \text{ for b} \neq 0, \gamma = 1; \\ -\exp\left(\alpha z\right) \text{ for } b = 0. \end{cases}$$

The functional form of the utility function



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- Constant or decreasing relative risk aversion:

$$\gamma'(z) \leq 0$$

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Group decision-making

- Consensus;

$$u(z)$$

- Compromise;

$$u(z) = \sum_{m=1}^{M} c_m u_m(z)$$

- Equal weighting:

$$u(z) = \frac{1}{M} \sum_{m=1}^{M} u_m(z)$$

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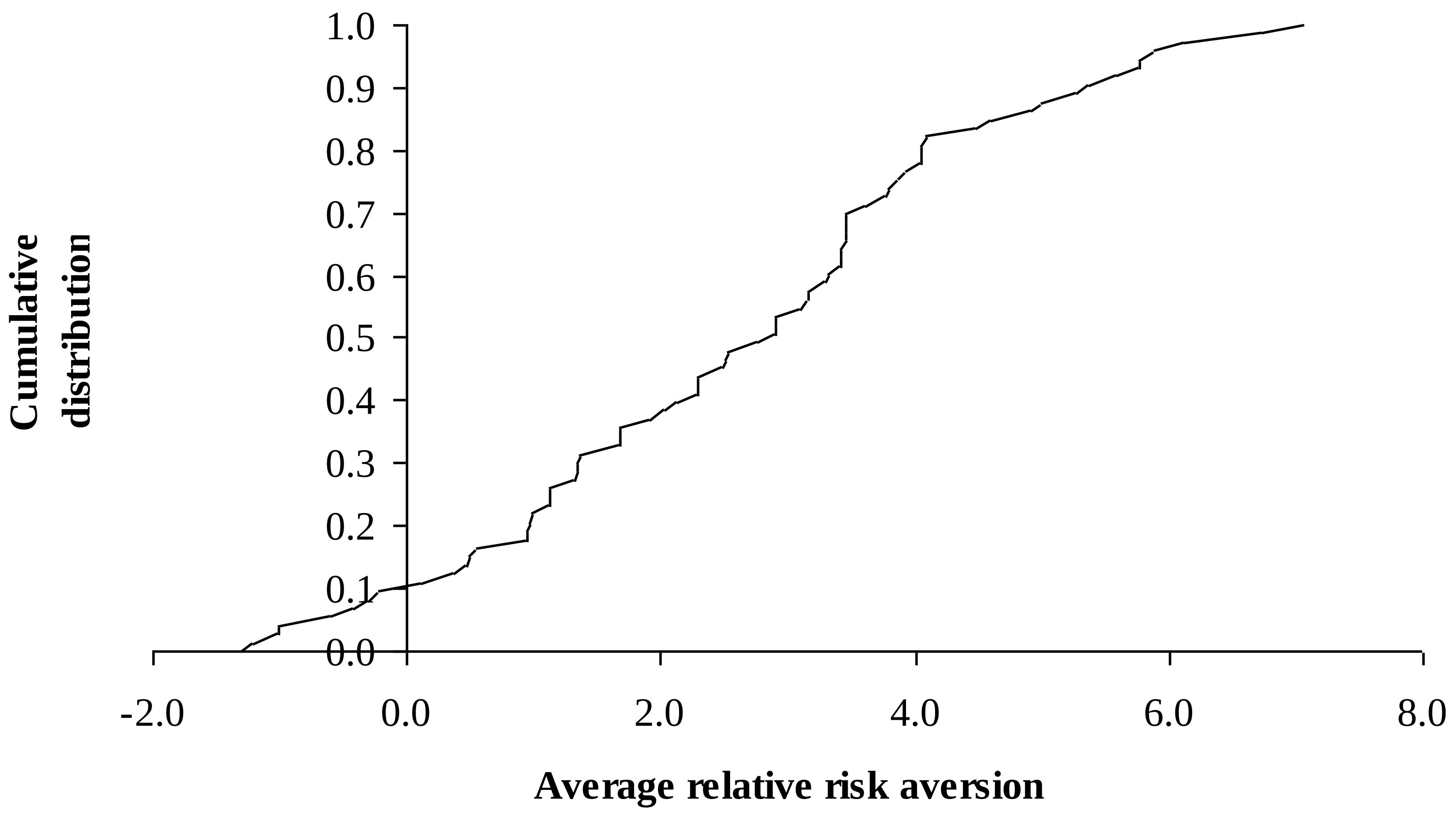
Levitan & Thomson



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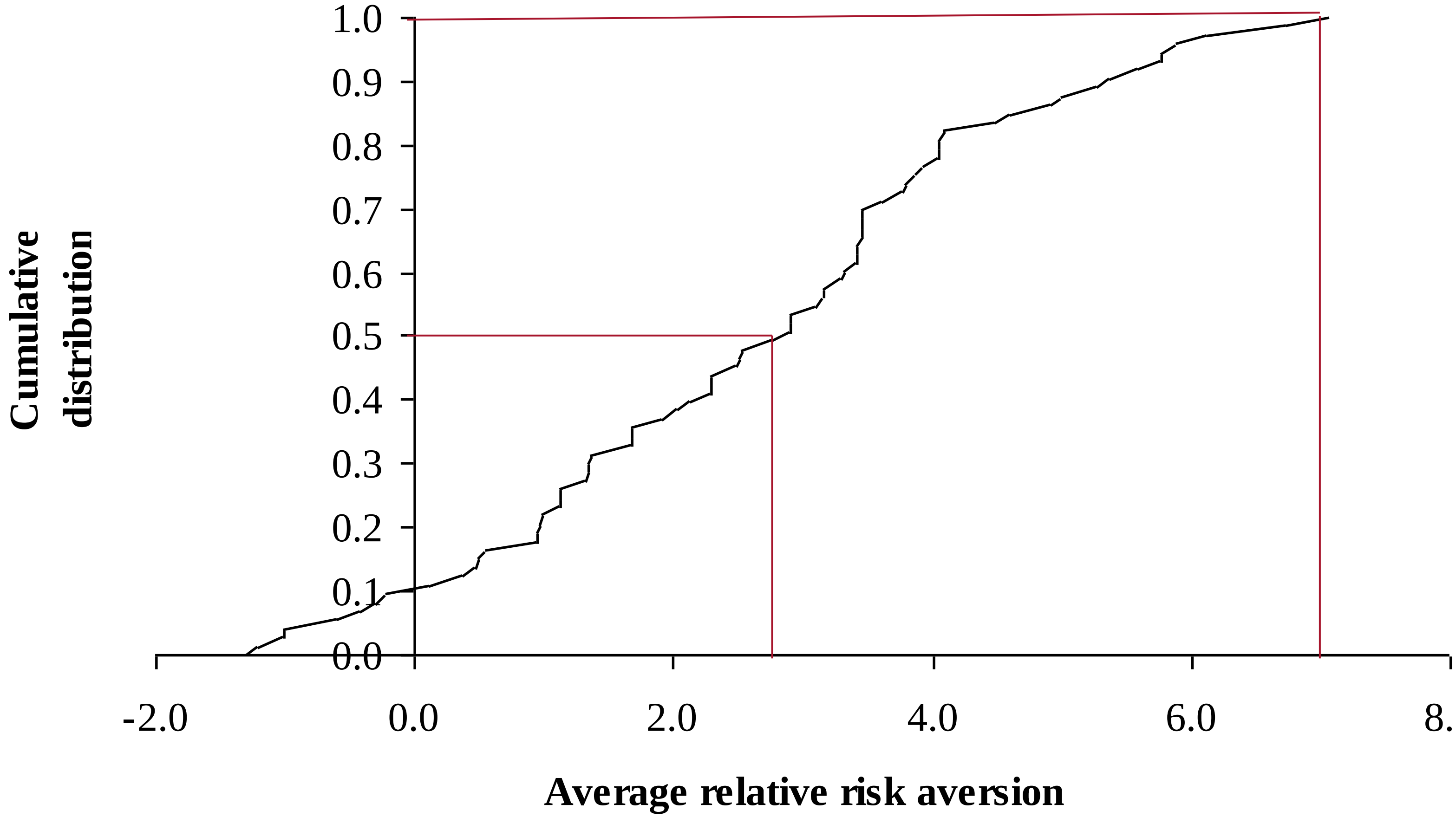

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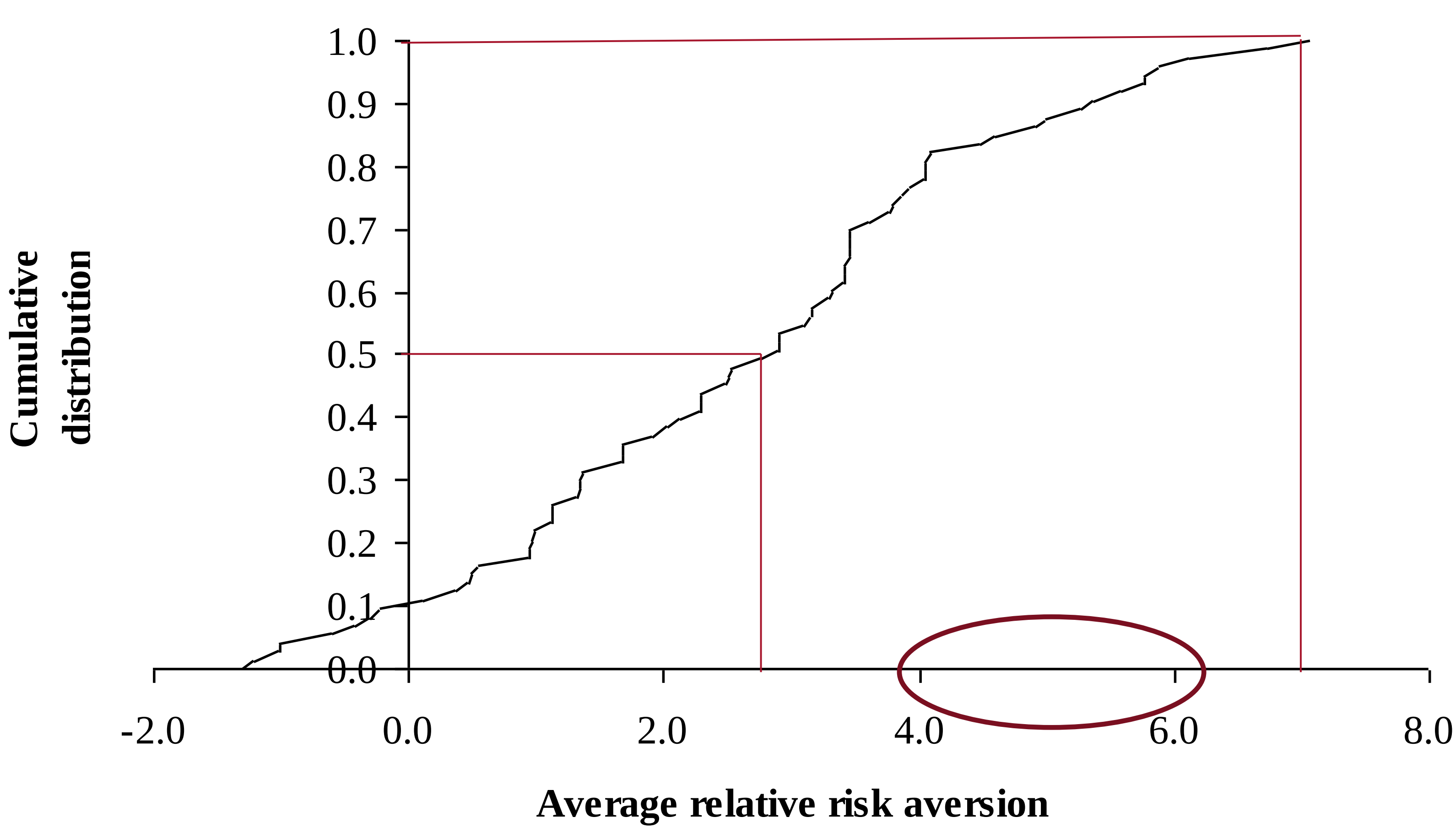

Levels of risk aversion and prudence

– Coefficient of relative risk aversion:

$$\gamma(z) = -z\frac{u''(z)}{u'(z)}$$

– Kimball's 'coefficient of relative prudence':

$$\pi(z) = -z\frac{u'''(z)}{u''(z)}$$

Levels of risk aversion and prudence

Criteria required:

1. coverage of the range of outcomes
2. continuity
3. unsatiation
4. relative risk aversion satisfactorily high
5. non-increasing relative risk aversion

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2. General review of literature
3. Definition of the utility function
4. **Decreasing relative risk aversion: the WARRA-class utility function**
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The WARRA-class utility function

$$u(z) = \frac{u_0(z) + c u_\infty(z)}{1+c}$$

where:

$$u_0(z) = \frac{z^{1-\gamma_0} - 1}{1-\gamma_0}$$

$$u_\infty(z) = \frac{z^{1-\gamma_\infty} - 1}{1-\gamma_\infty}$$

$$c > 0$$

$$\gamma_0 \geq \gamma_\infty > 1$$

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$$c > 0$$

$$\gamma_0 \geq \gamma_\infty > 1$$

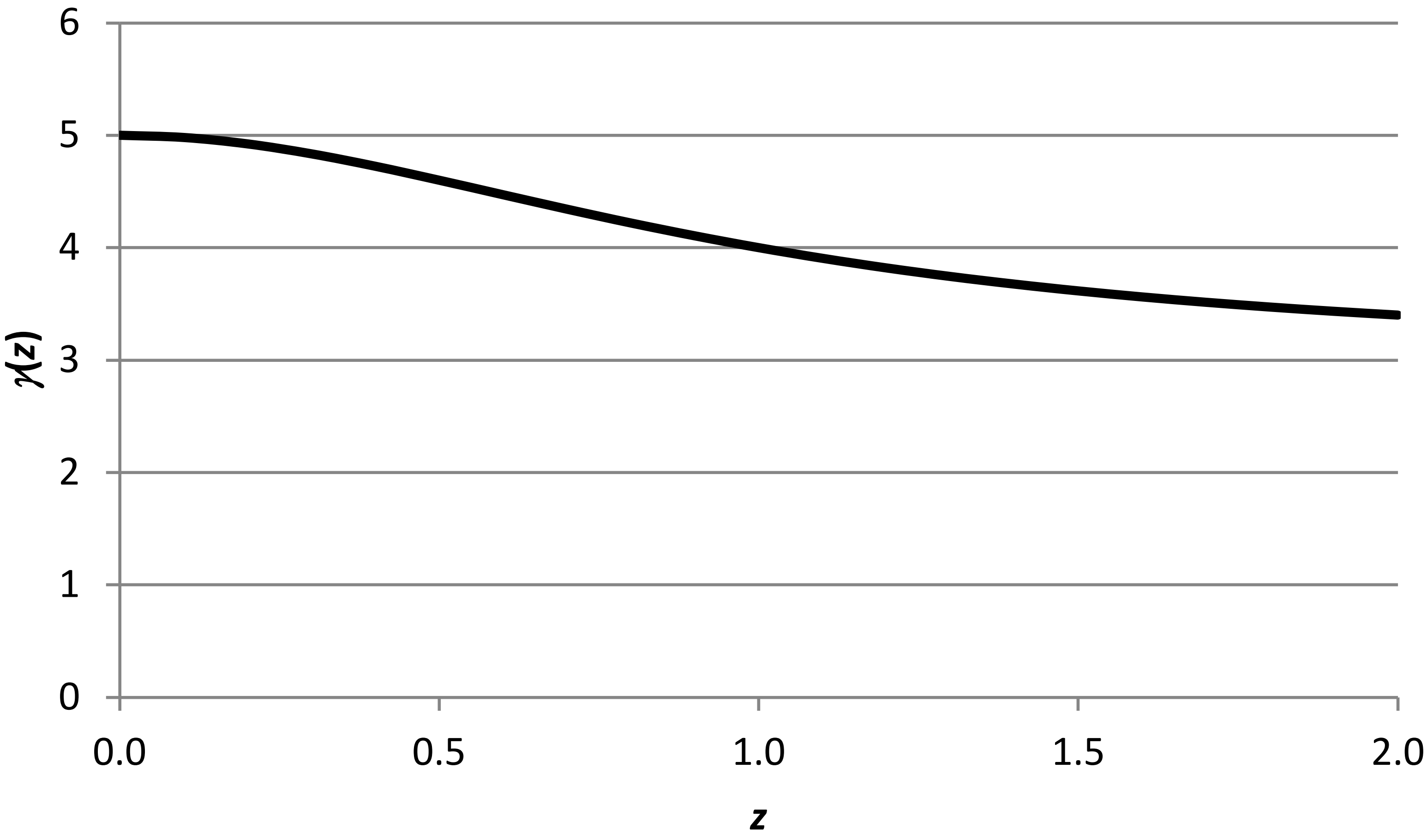
WARRA-class relative risk aversion

$$\gamma(z) = \frac{\gamma_0 + c\gamma_\infty z^\lambda}{1 + cz^\lambda};$$

where:

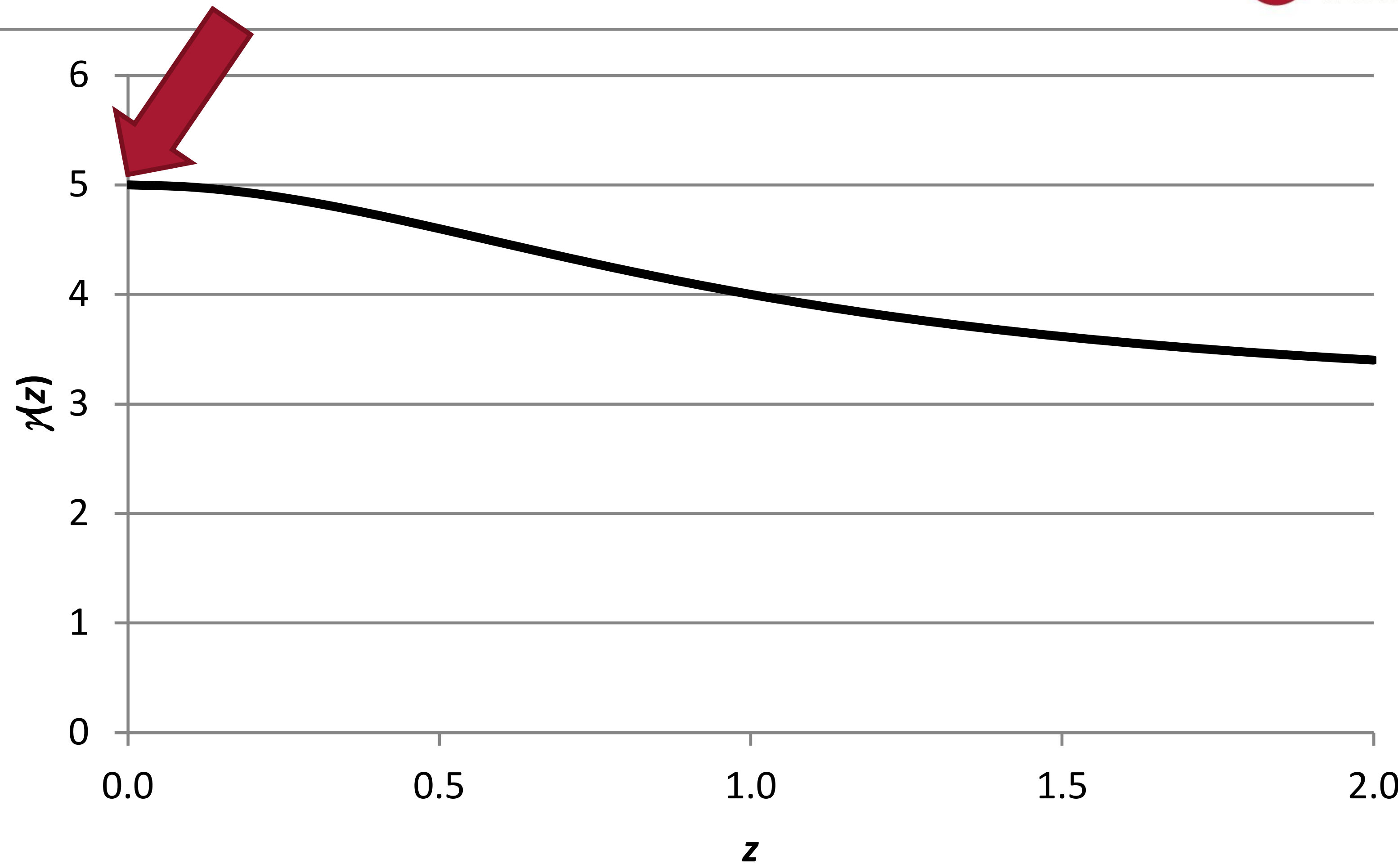
$$\lambda = \gamma_0 - \gamma_\infty$$

WARRA-class relative risk aversion



WARRA-class relative risk aversion

ACTUARIAL SOCIETY OF SOUTH AFRICA



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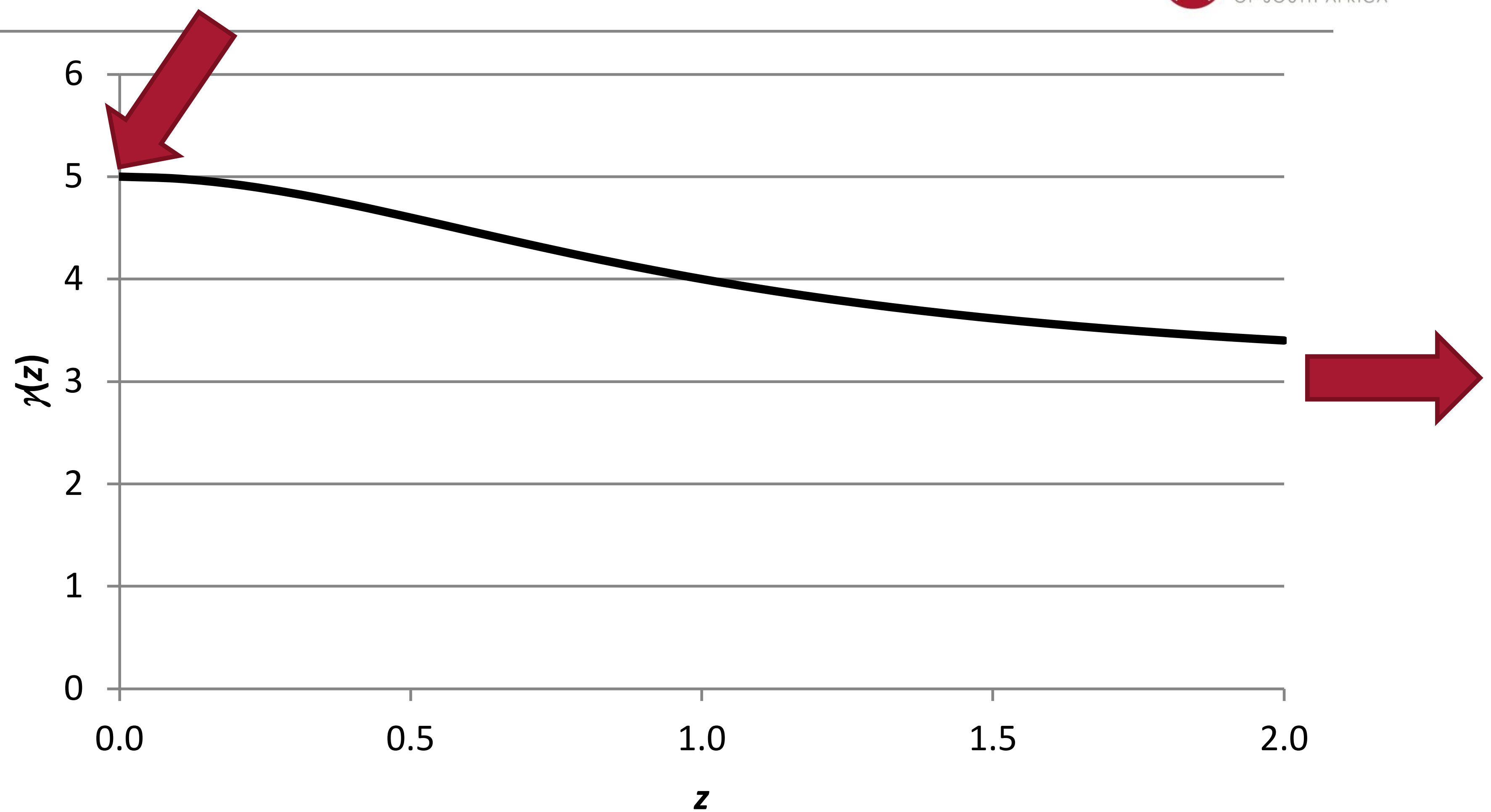

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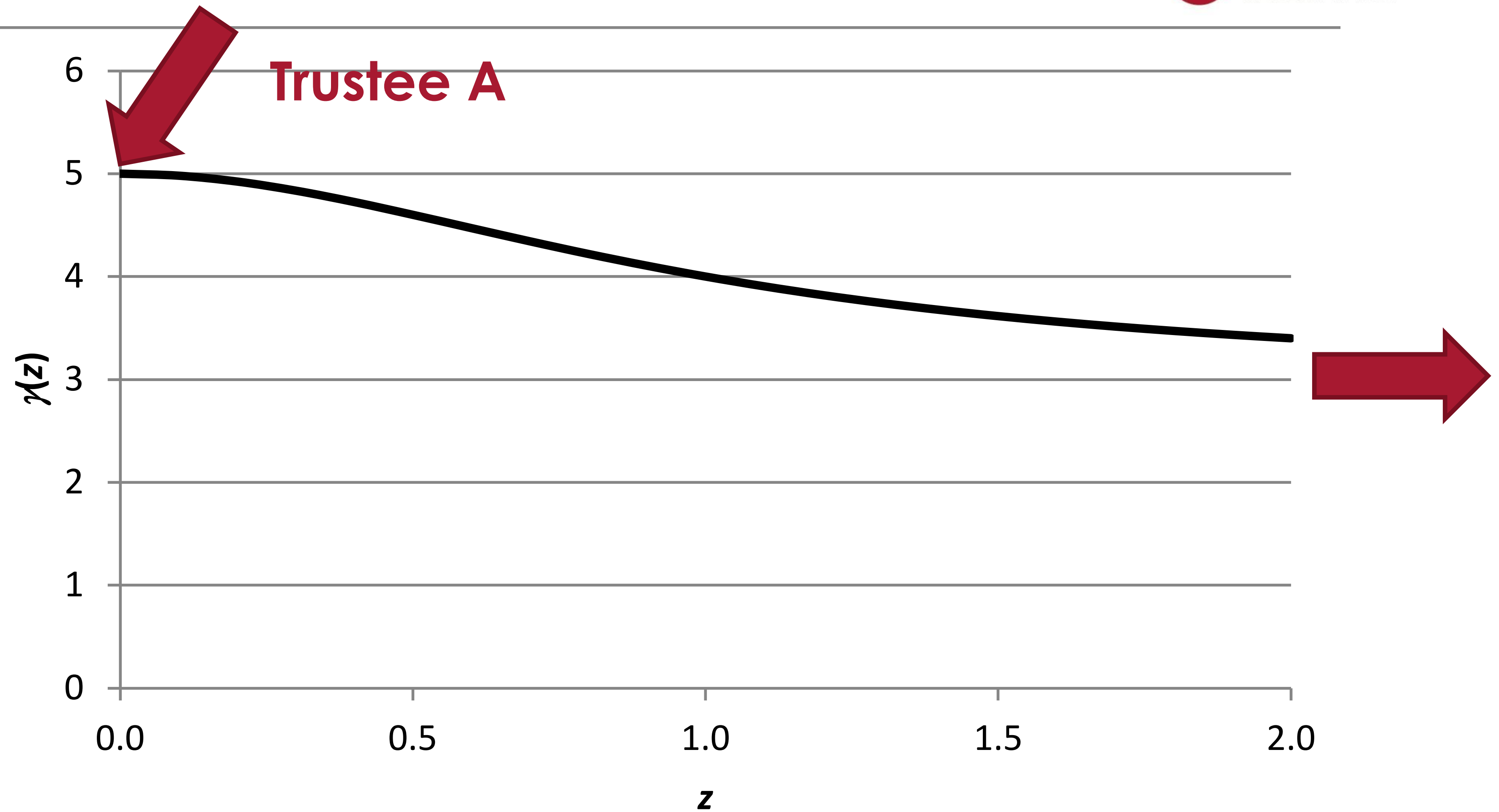

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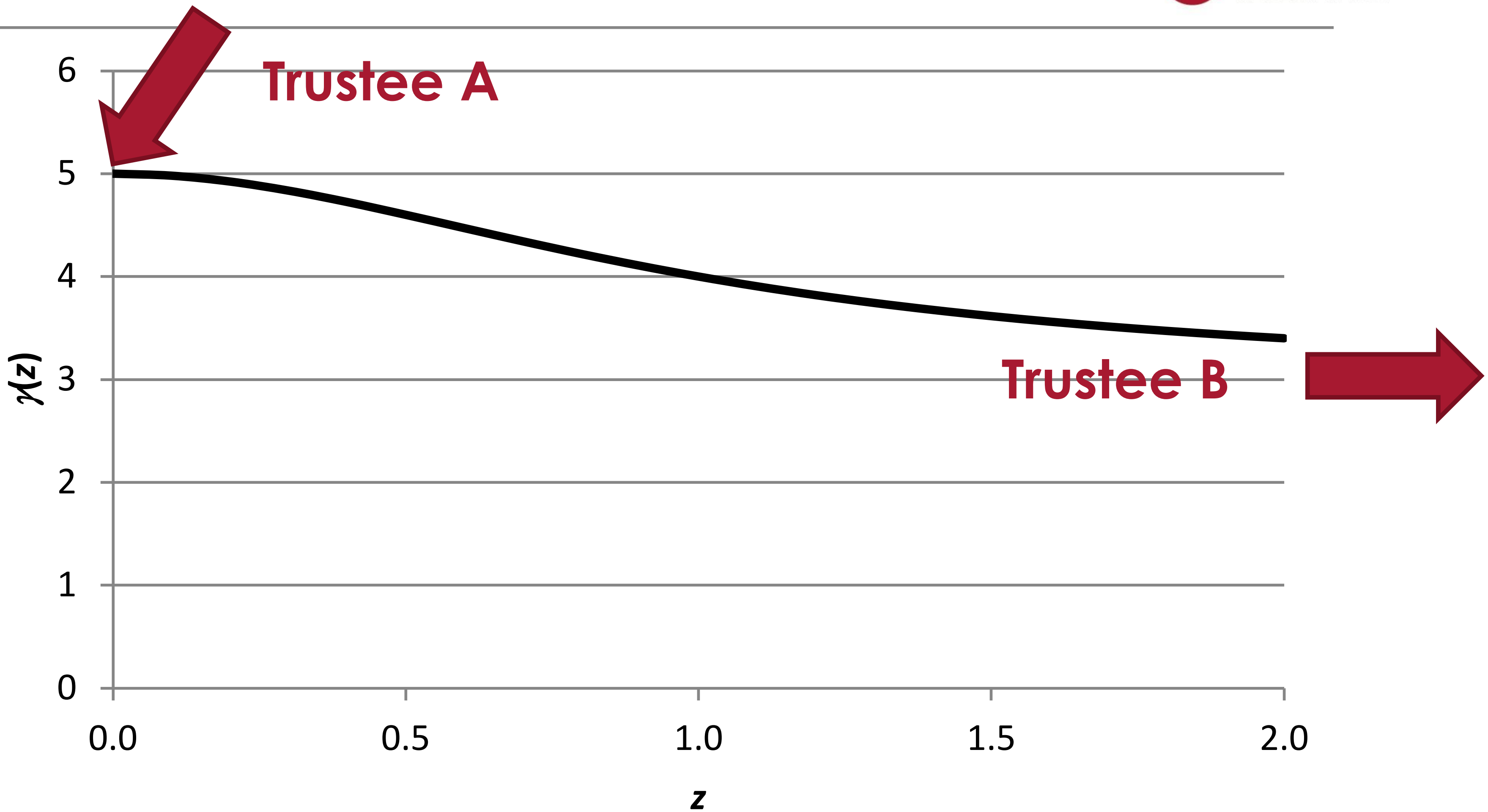
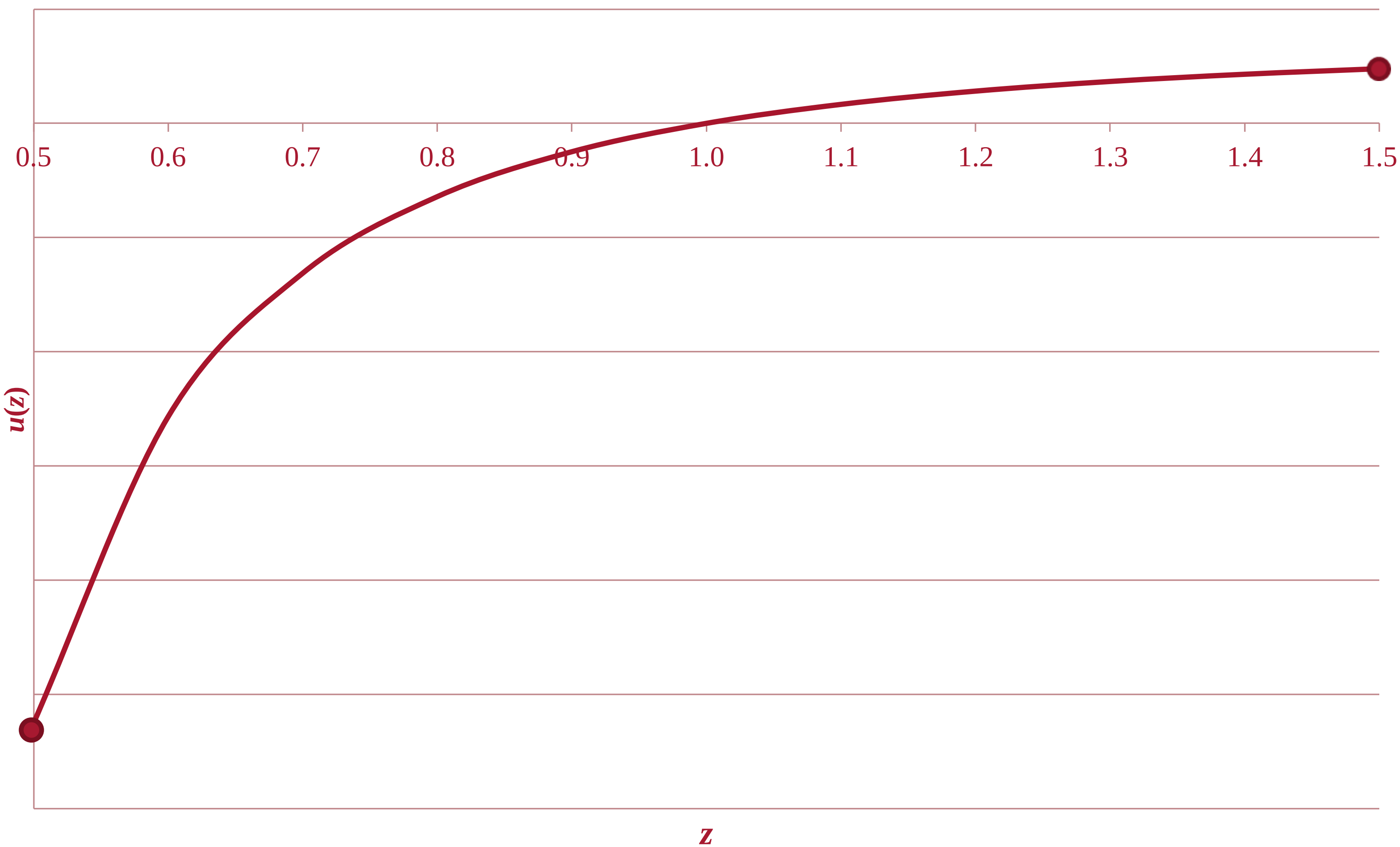


Parameterising a WARRA-class utility function



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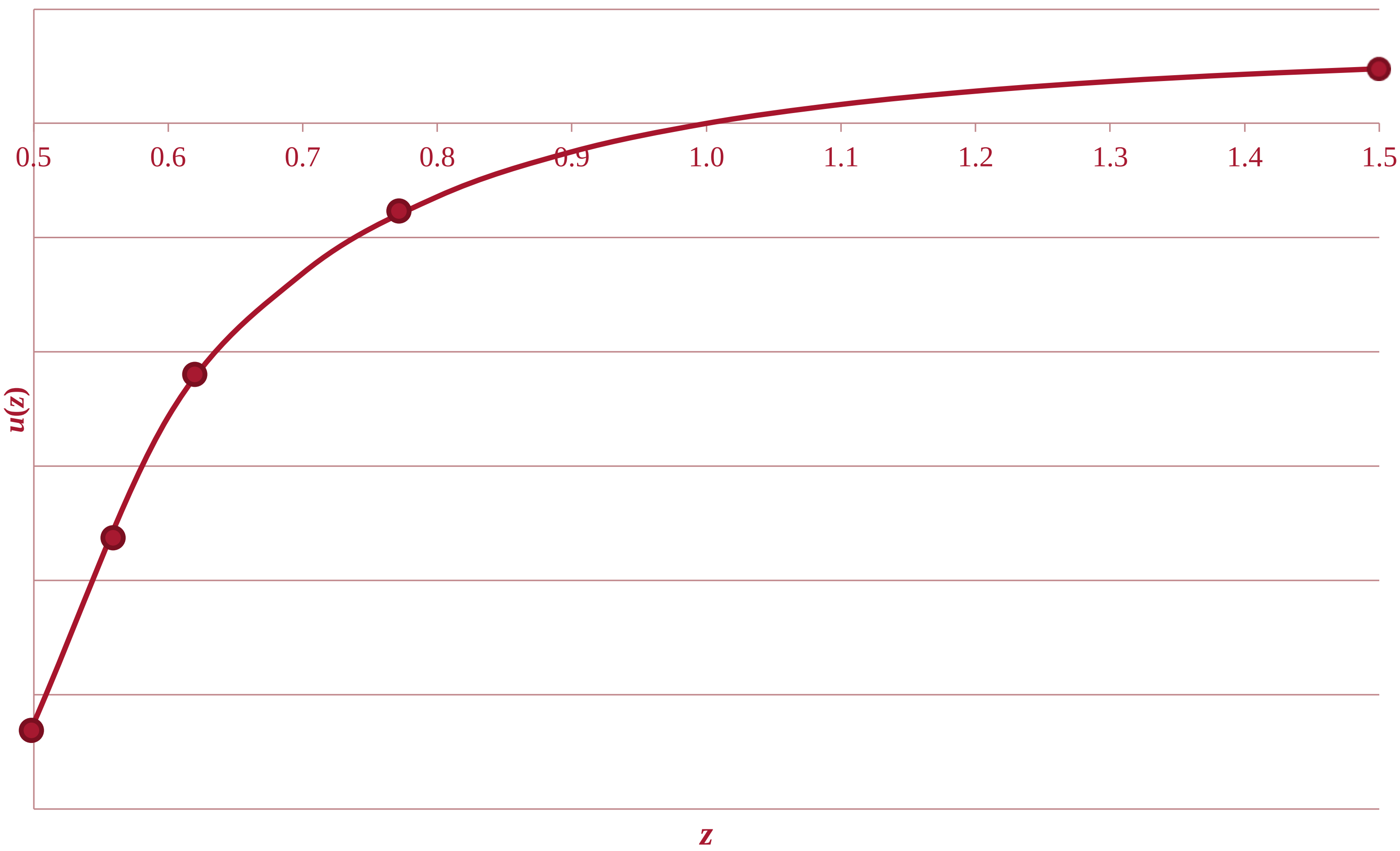

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3. Definition of the utility function

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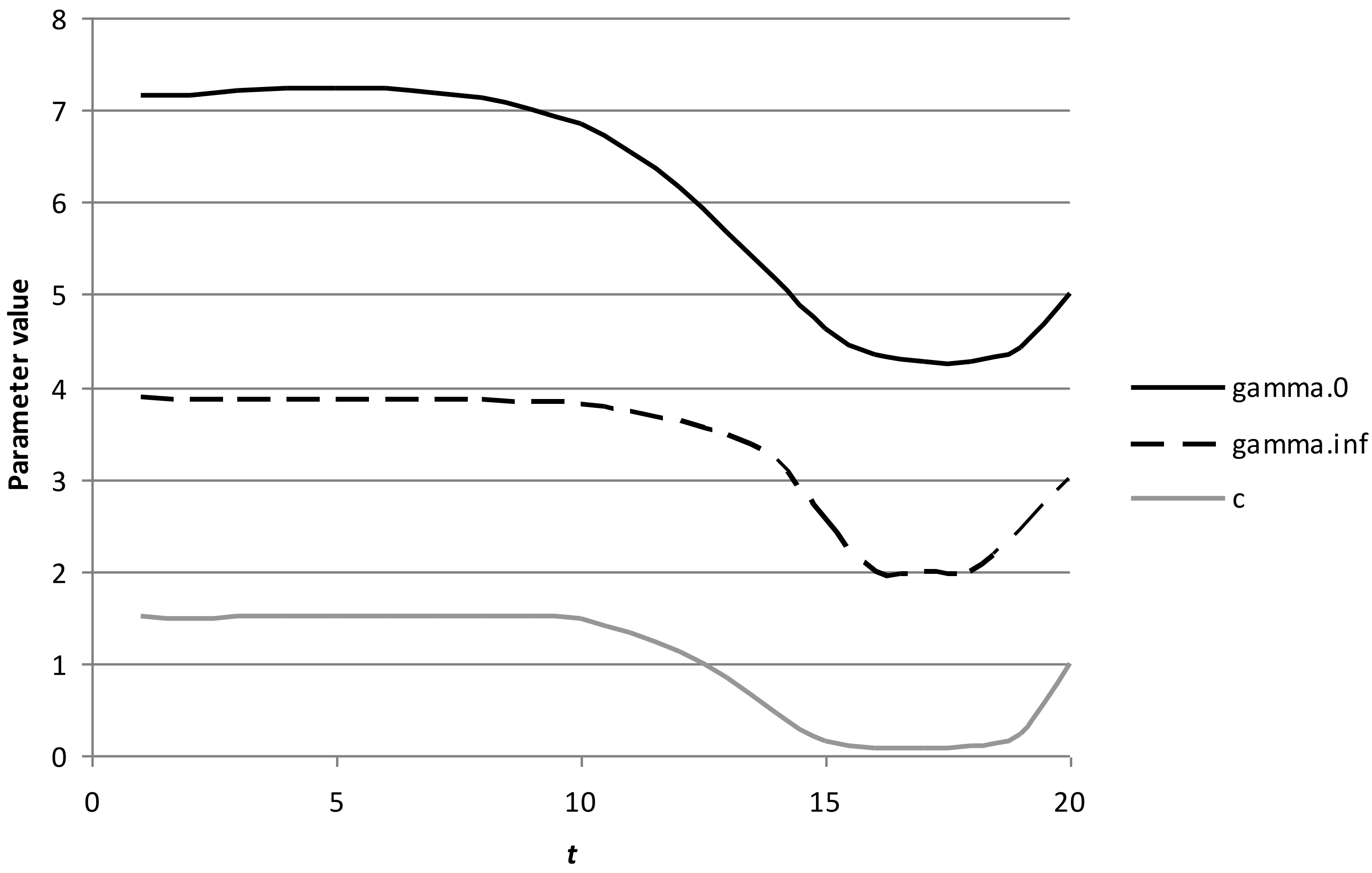
Illustrative results: parameters of the indirect utility functions



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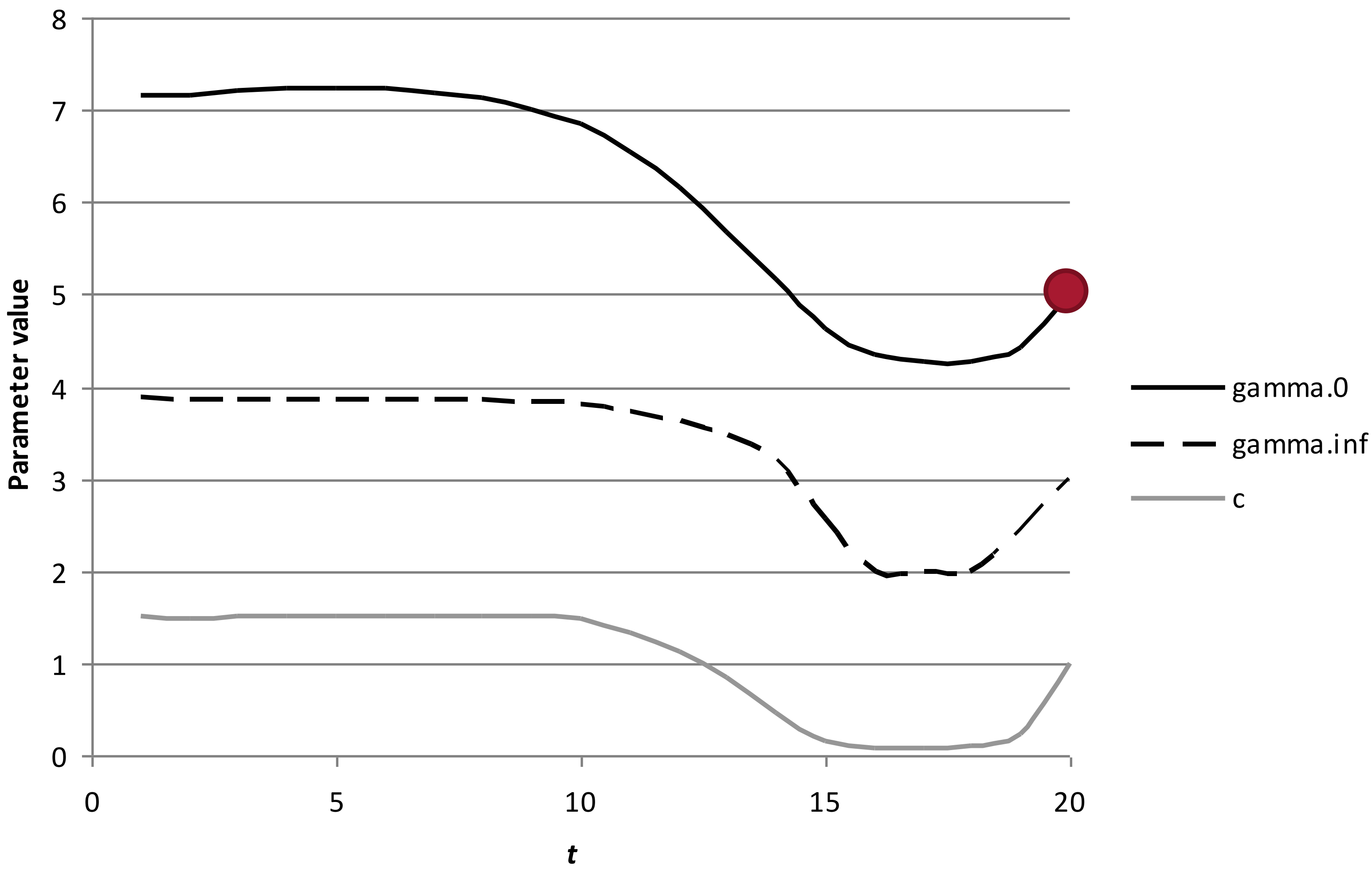

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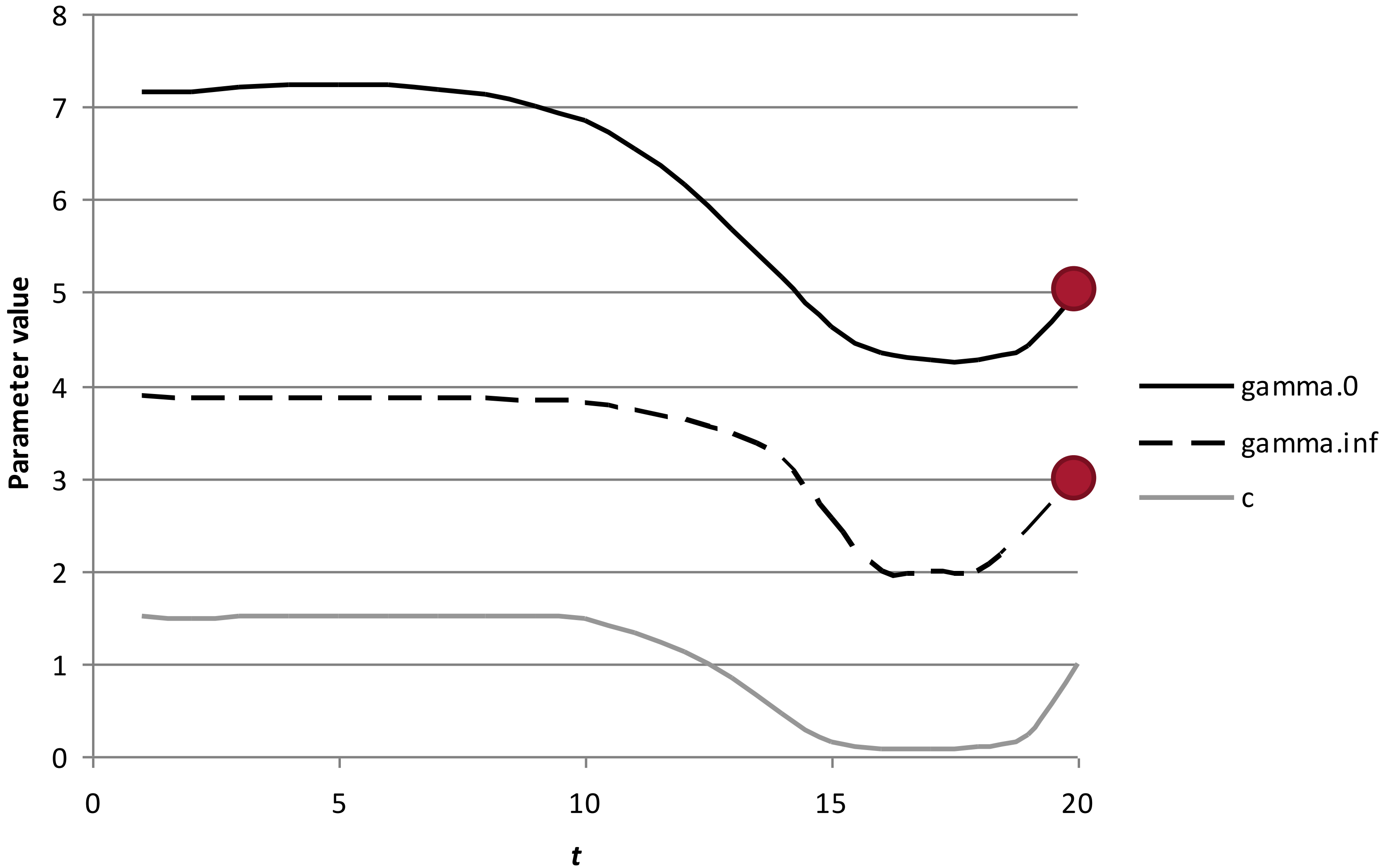

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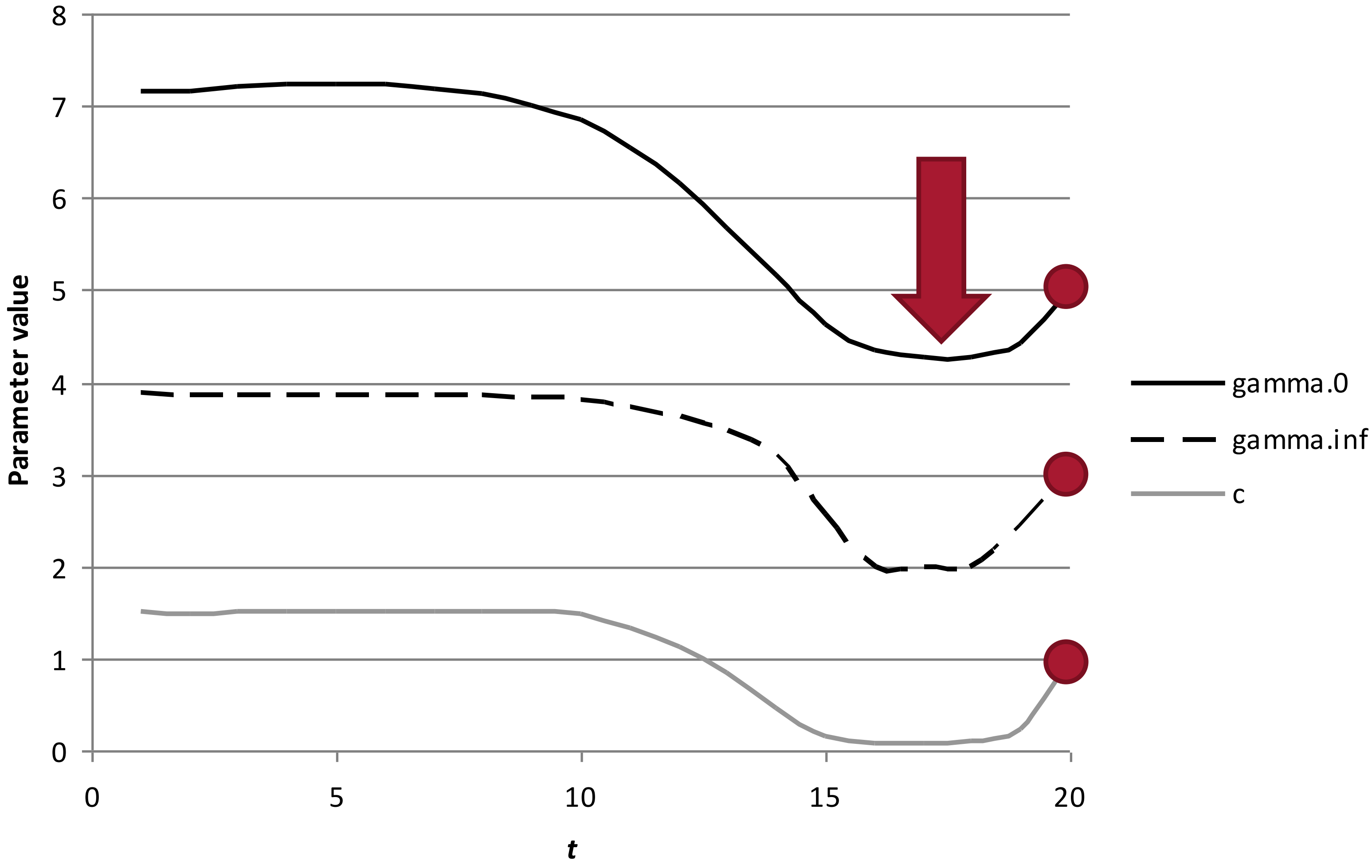

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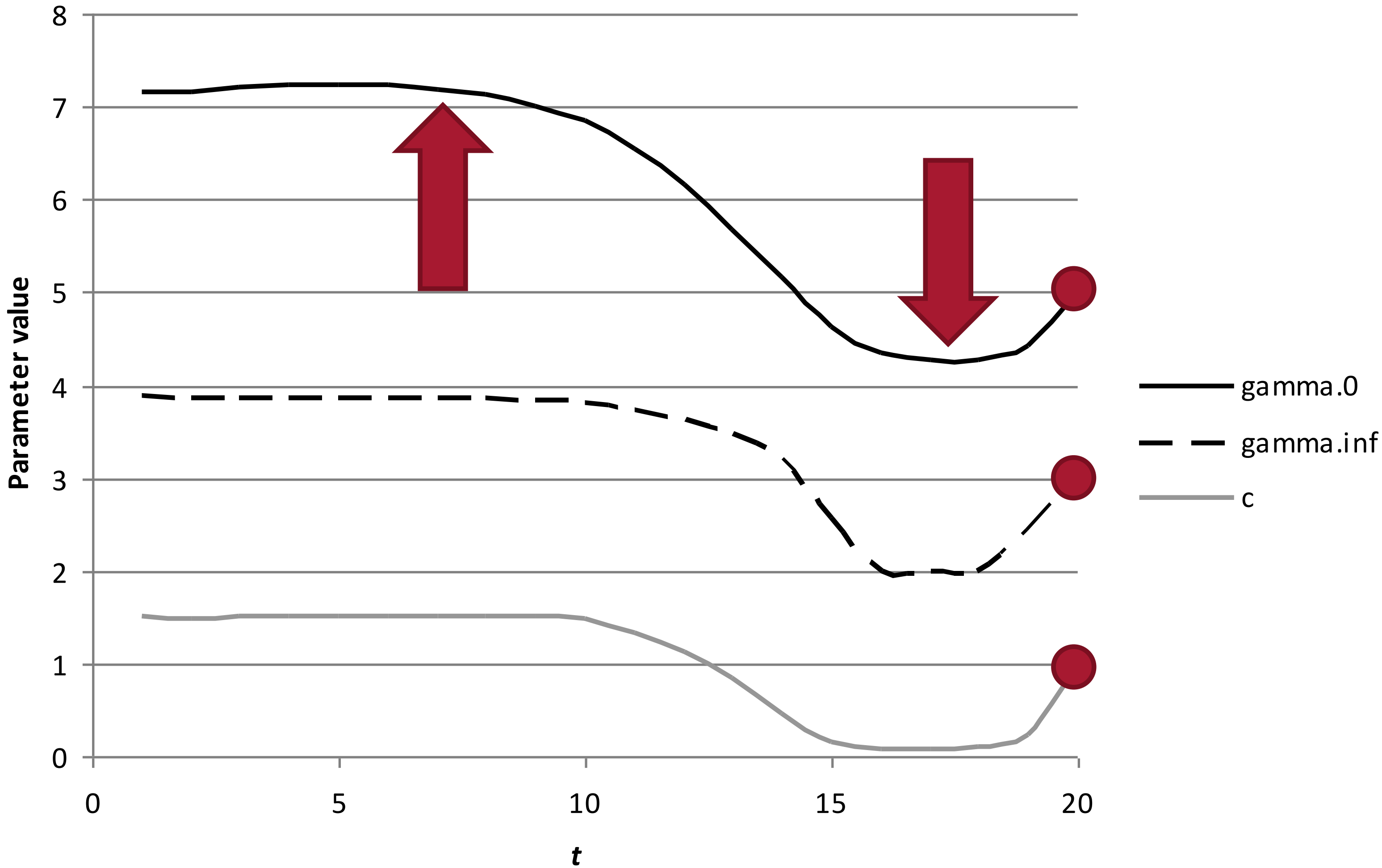

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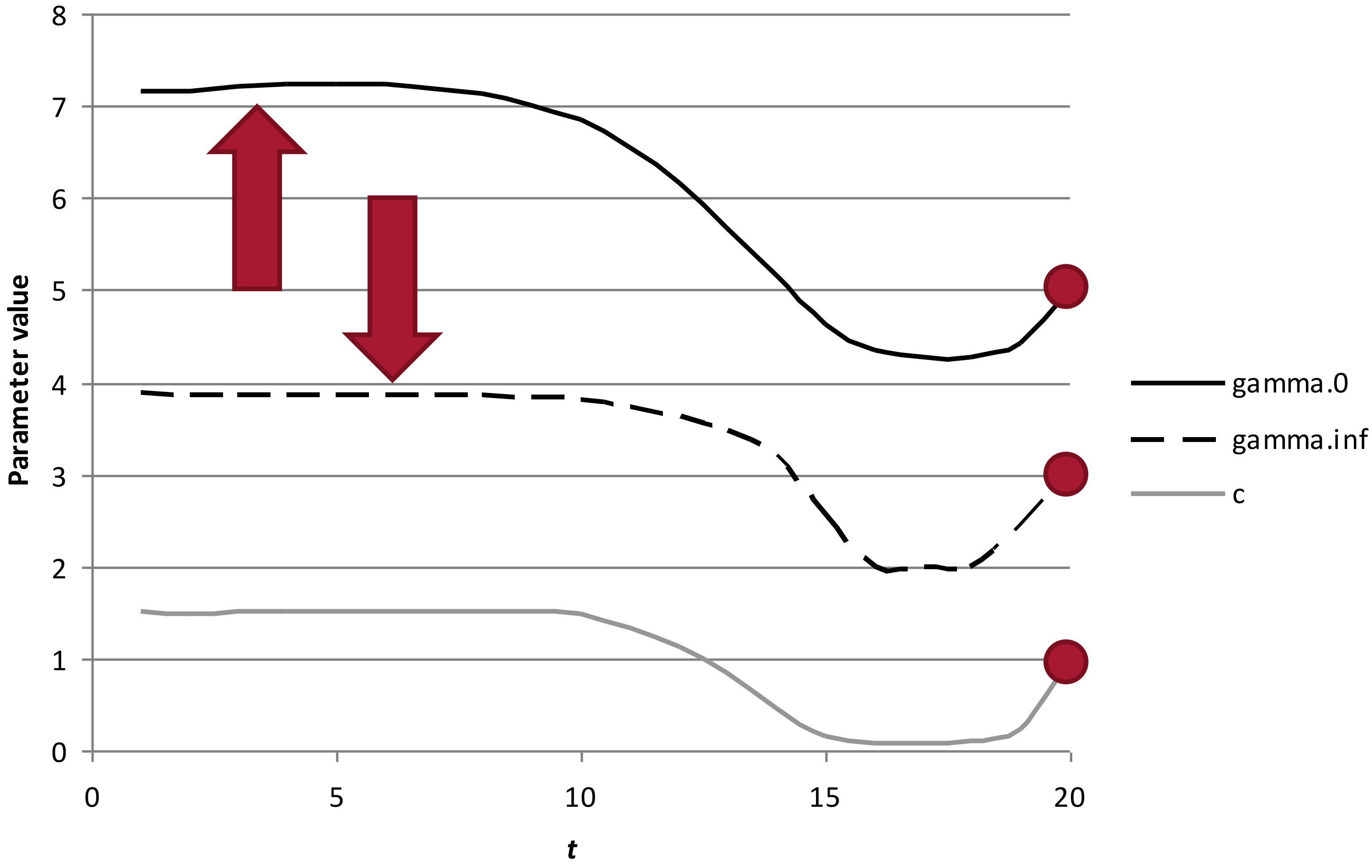
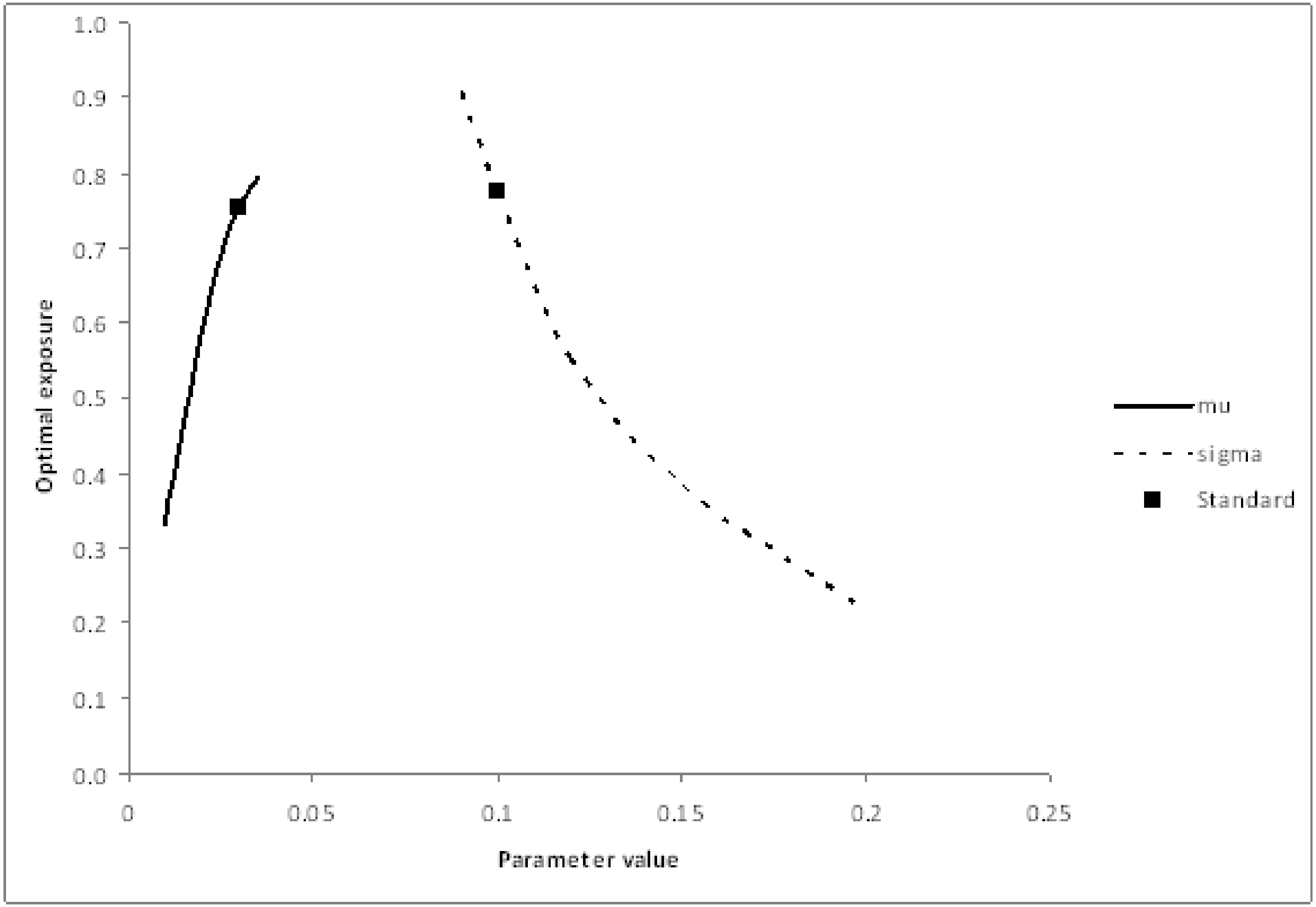


Illustrative results

- Optimal exposure to the risky asset: 75,4%

Illustrative results: parameters of the model of risky assets



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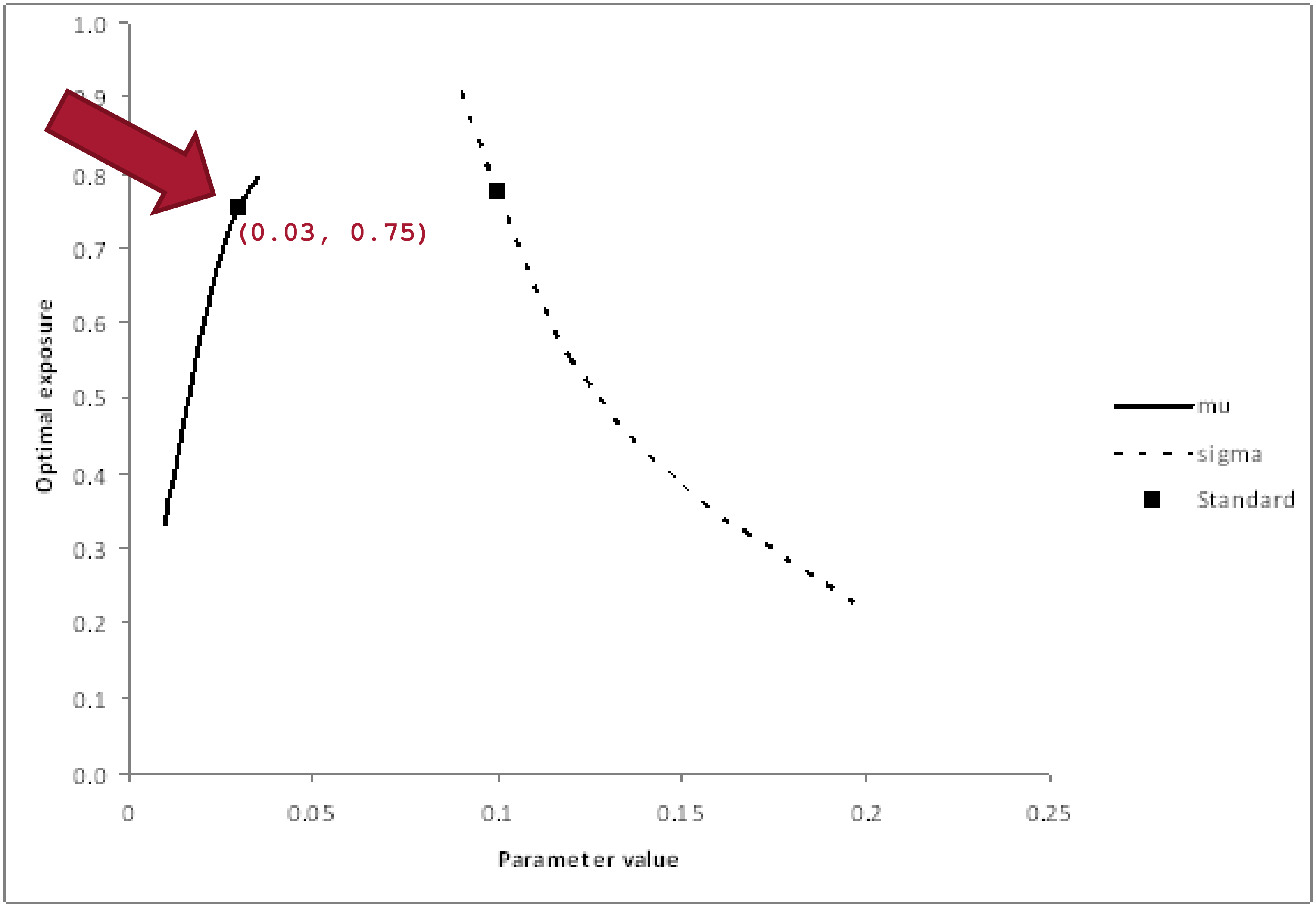

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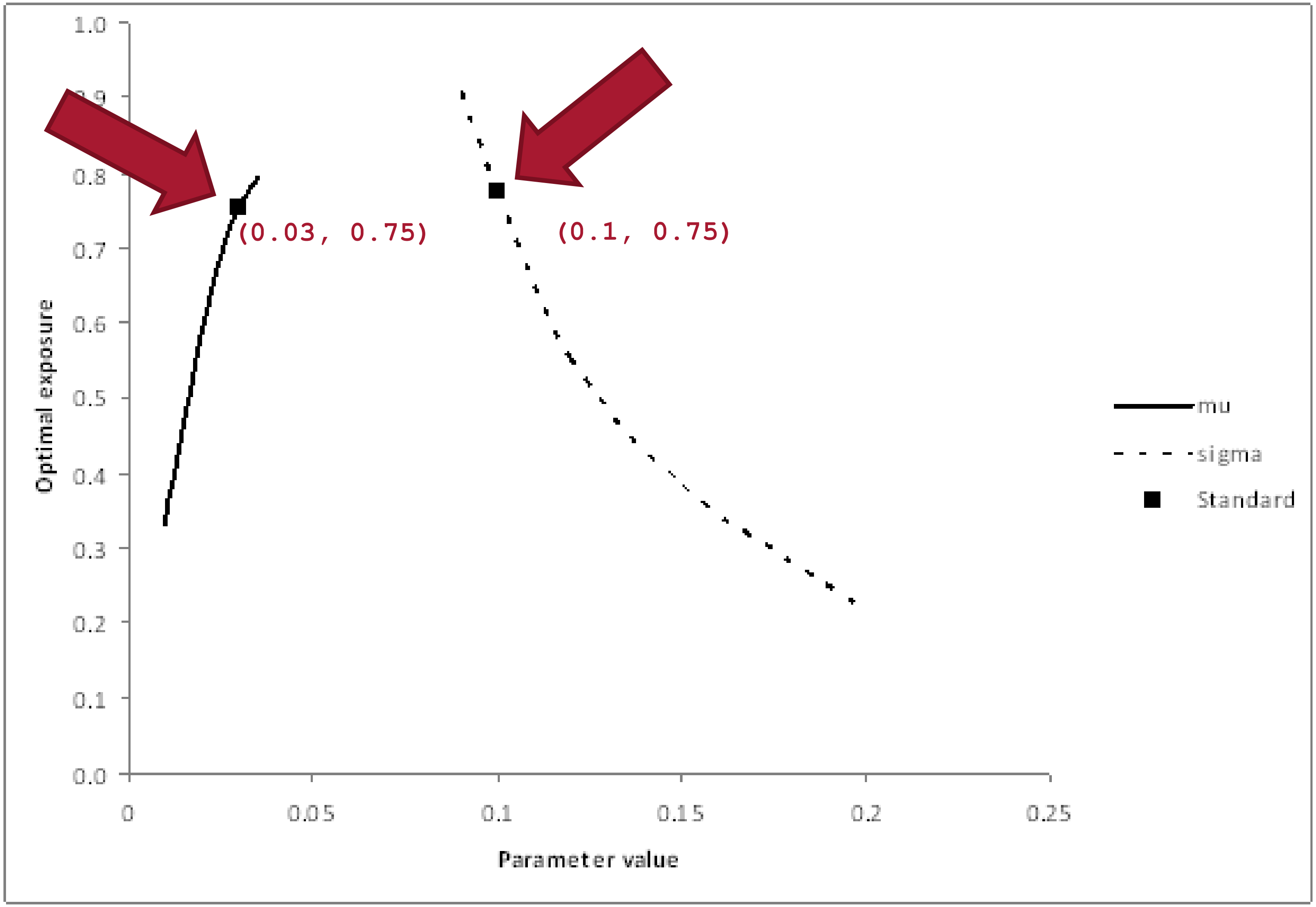

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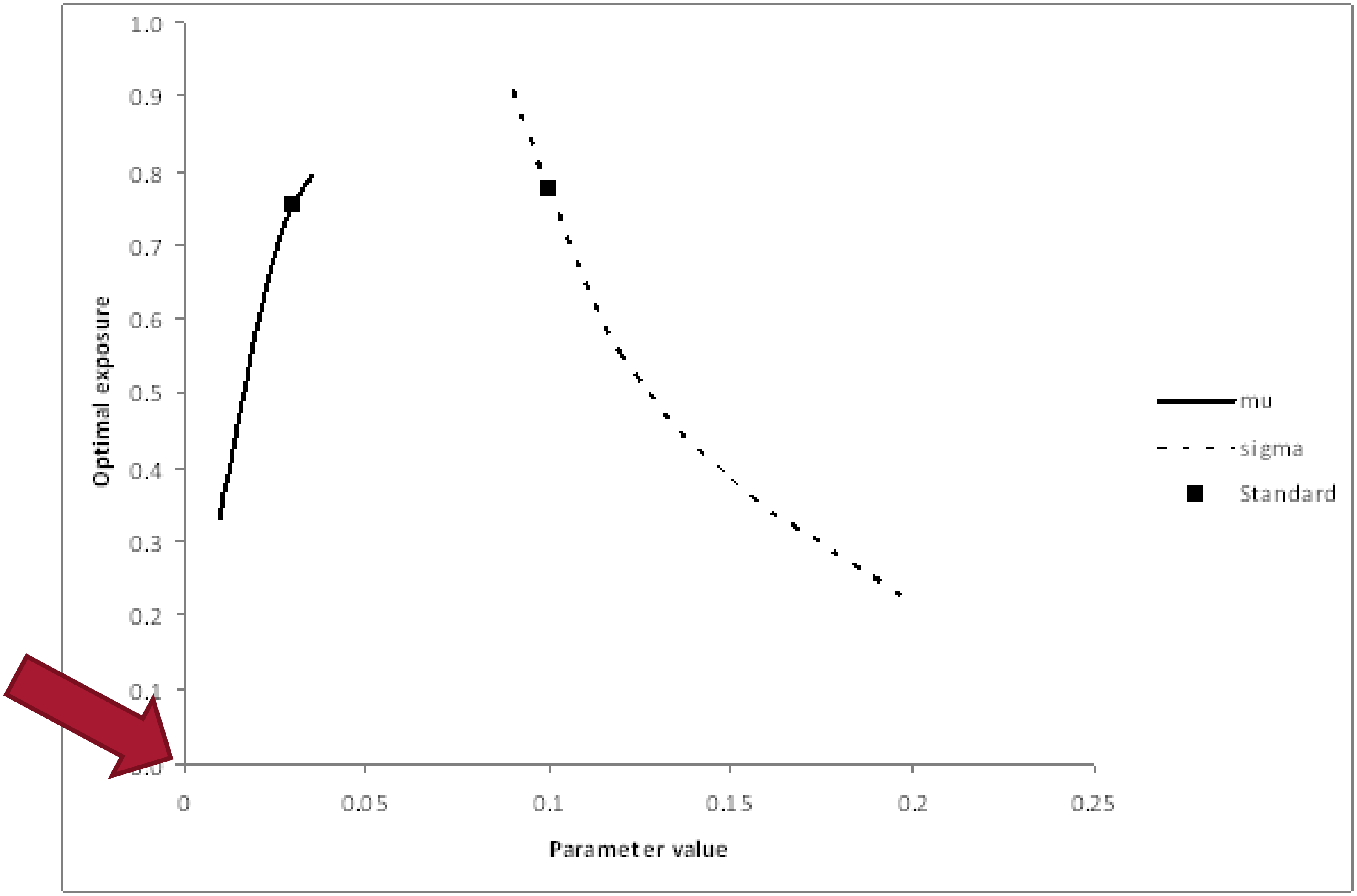

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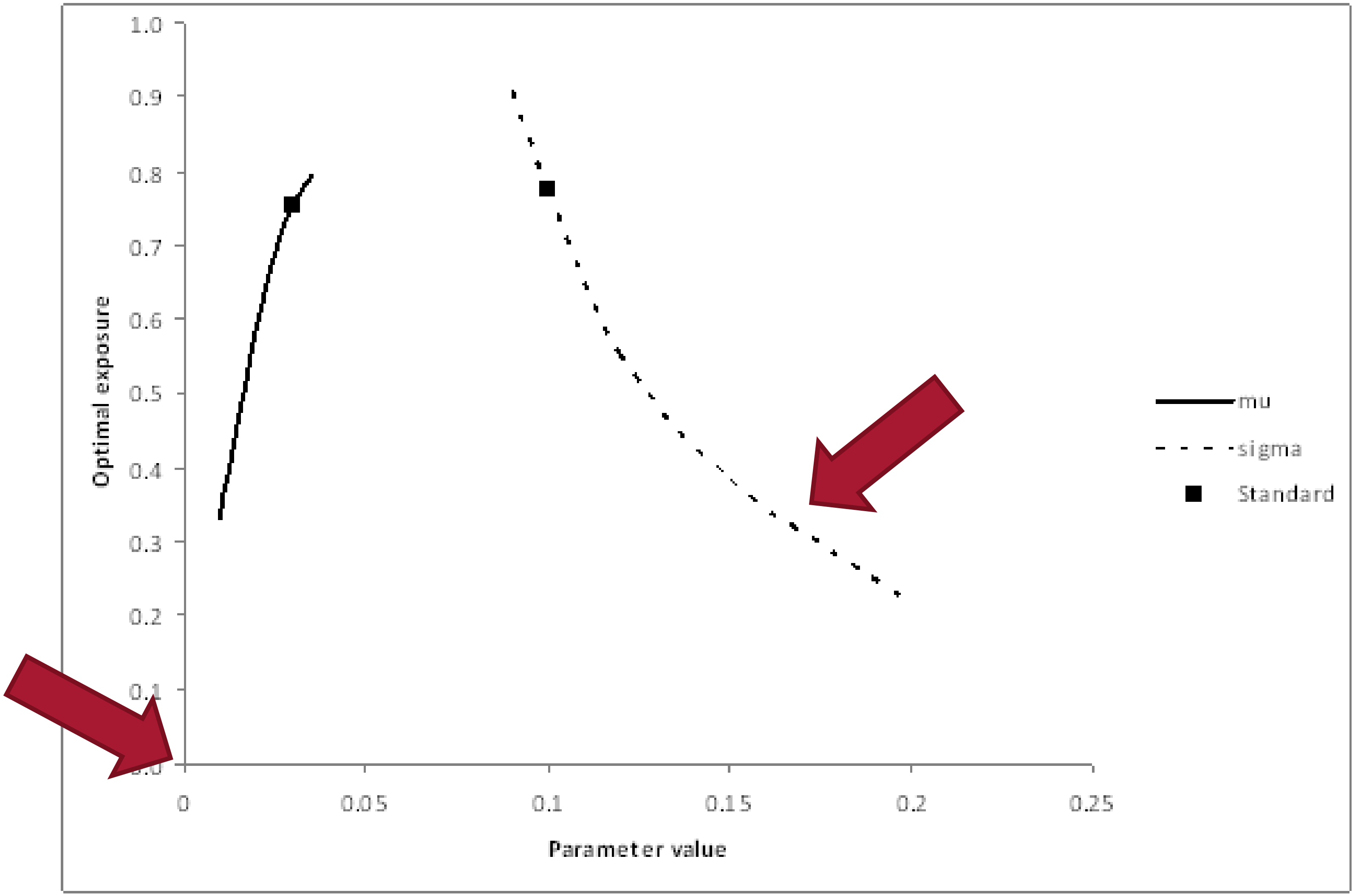

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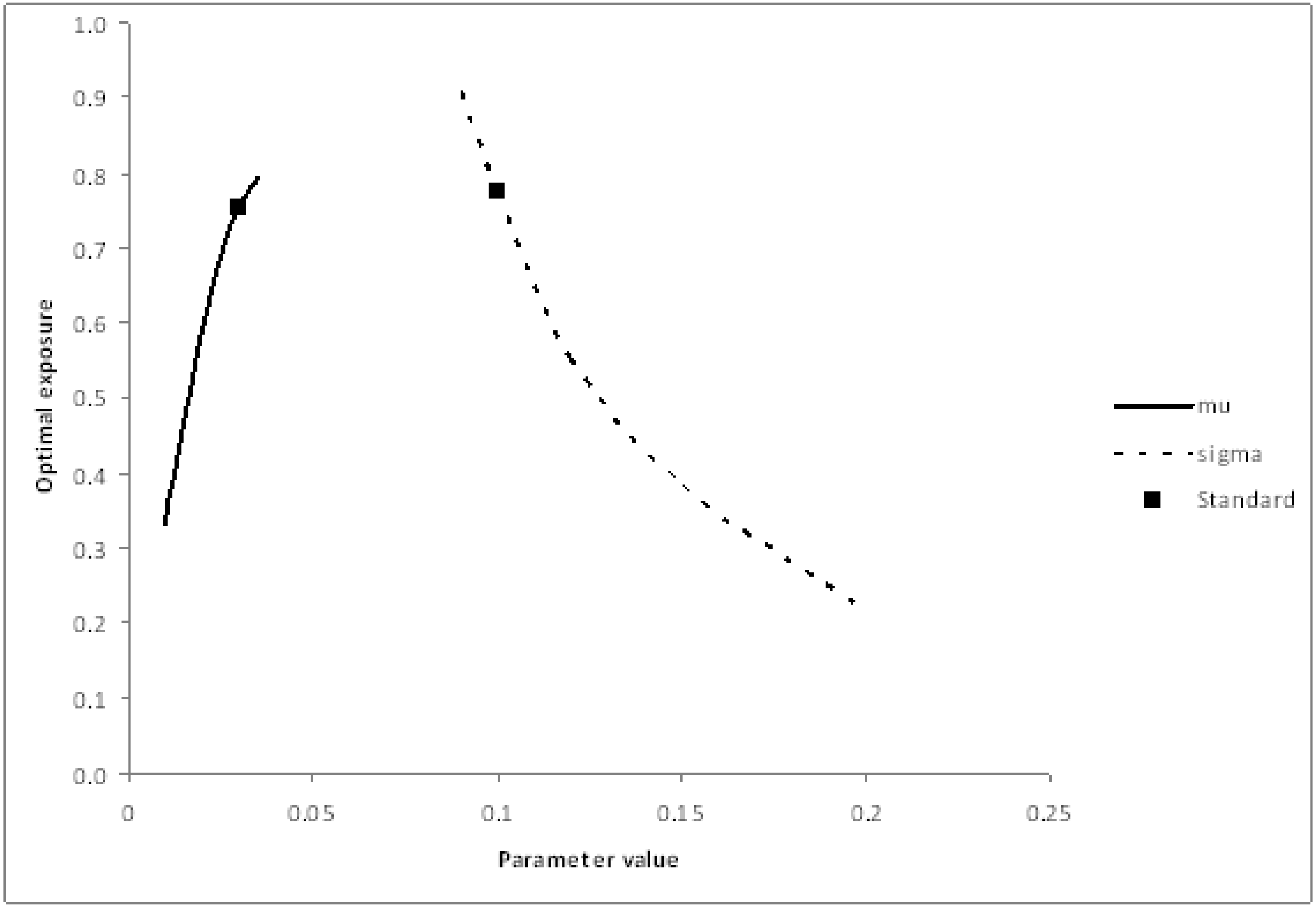

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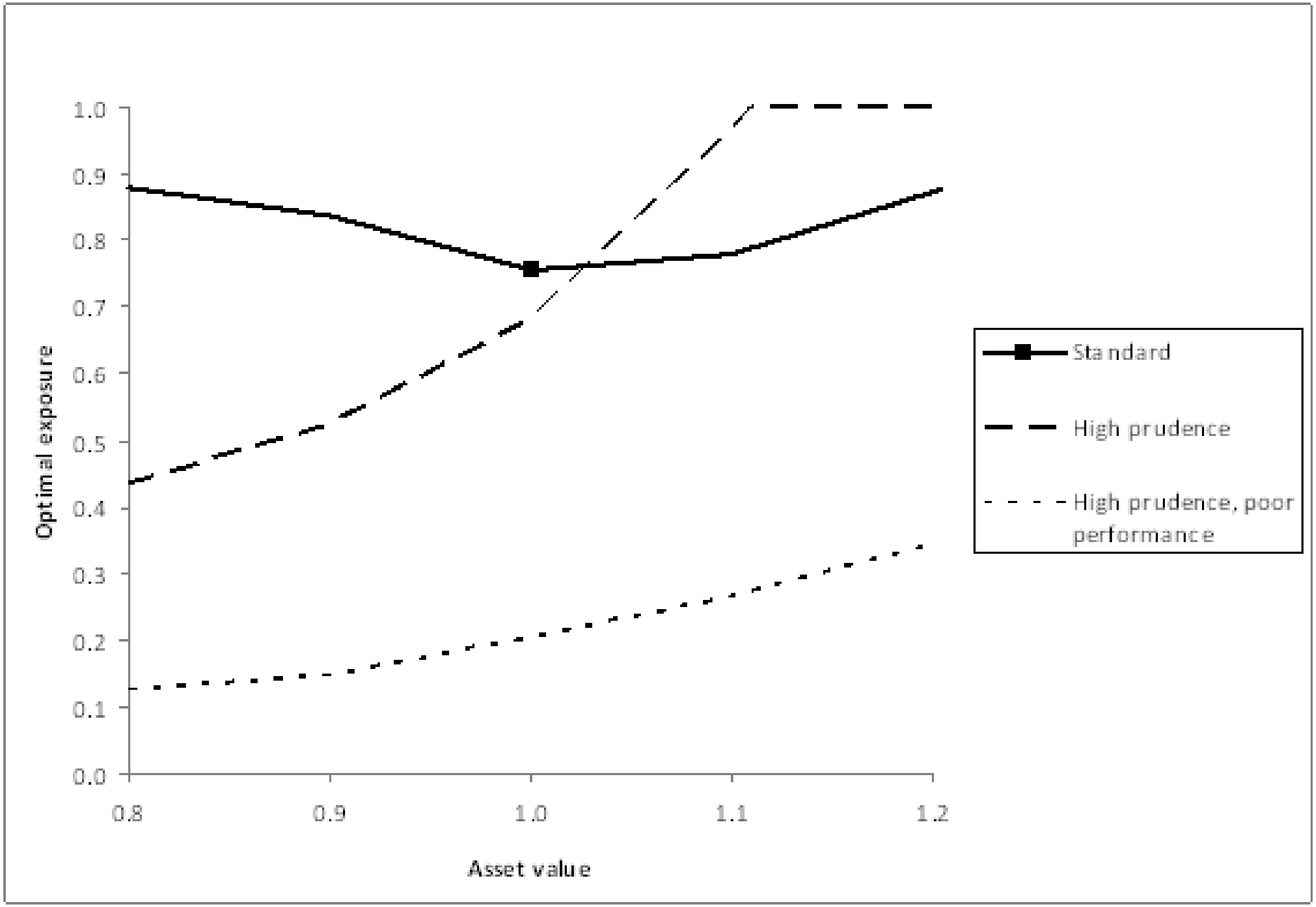

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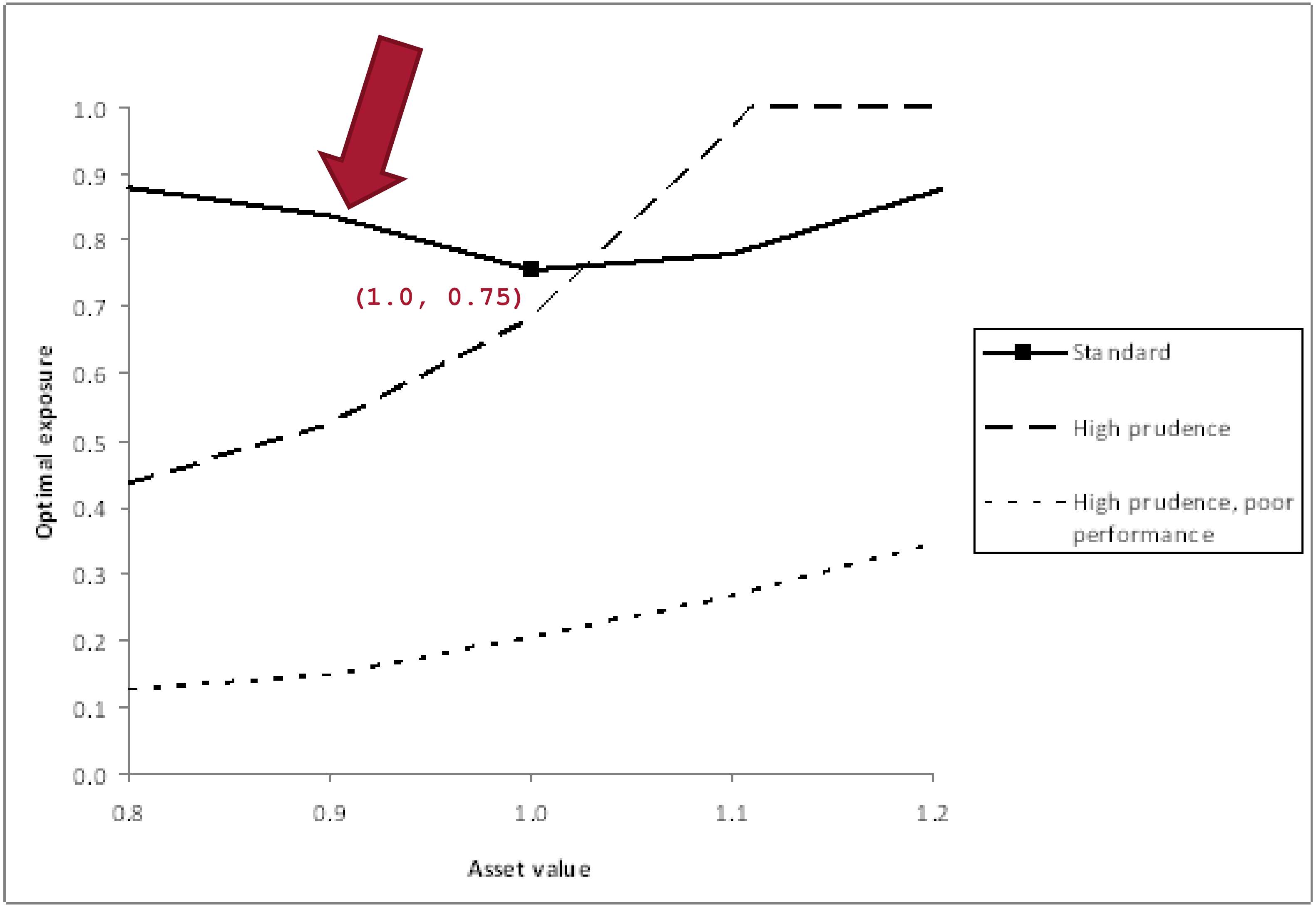

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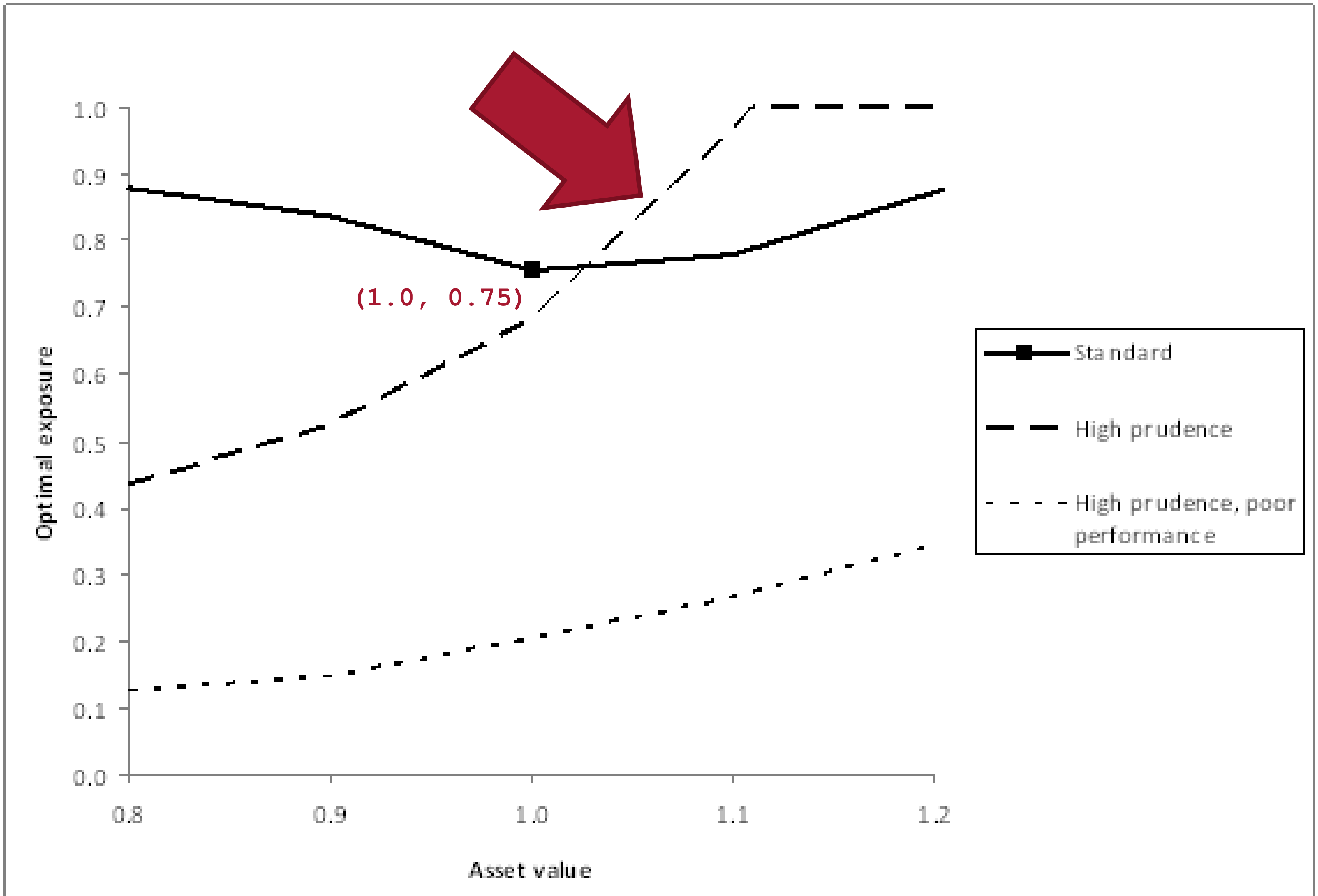

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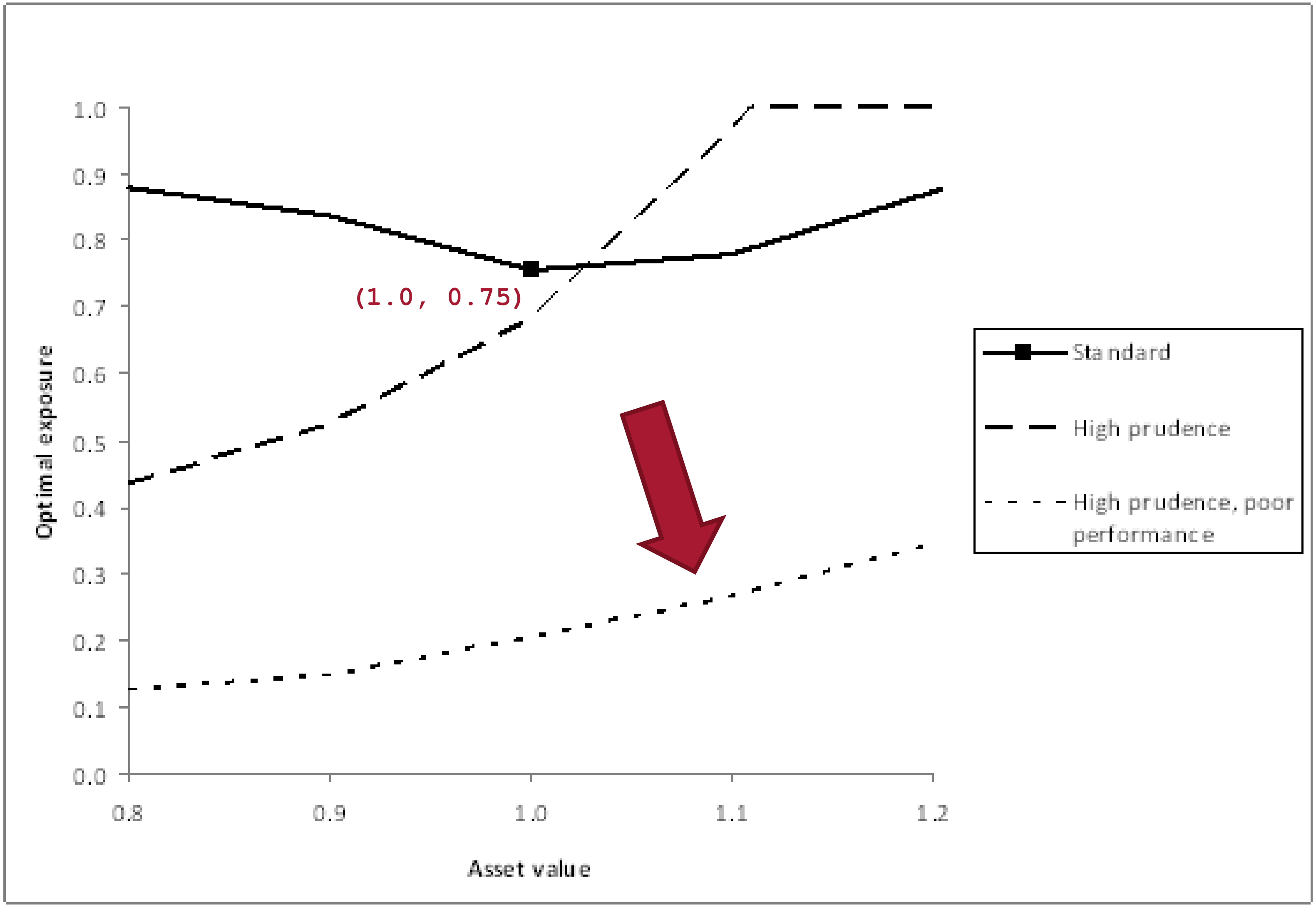

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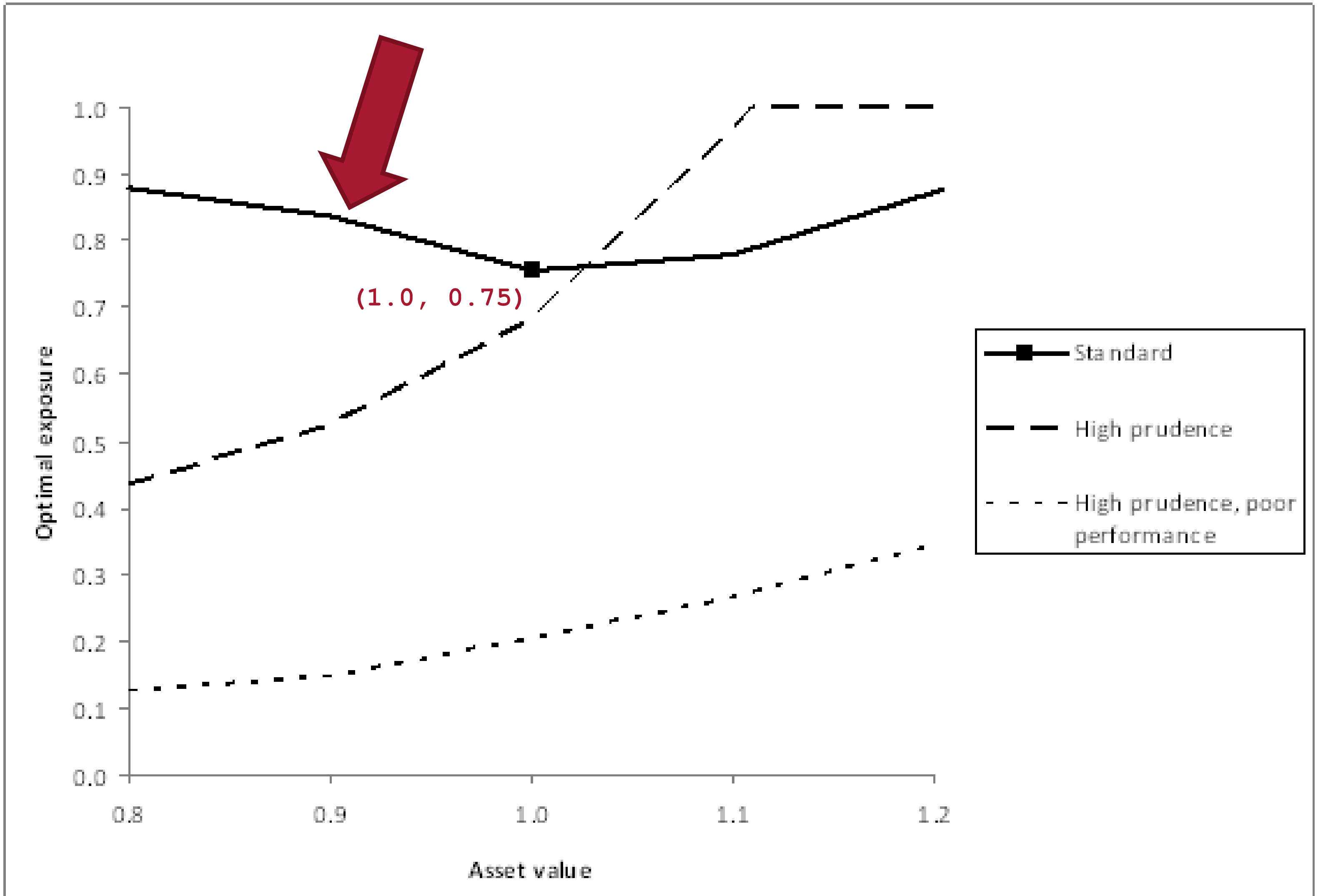

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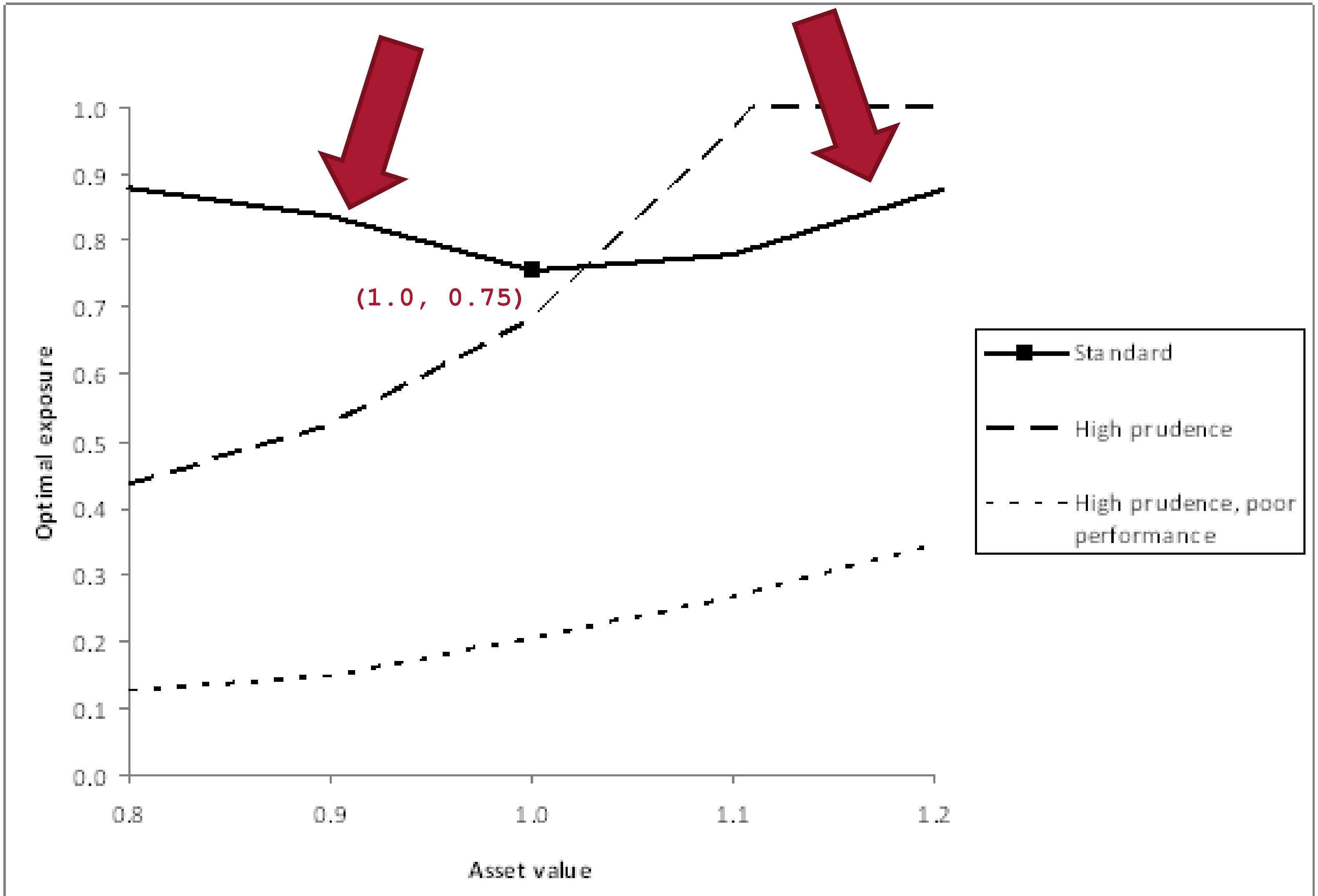

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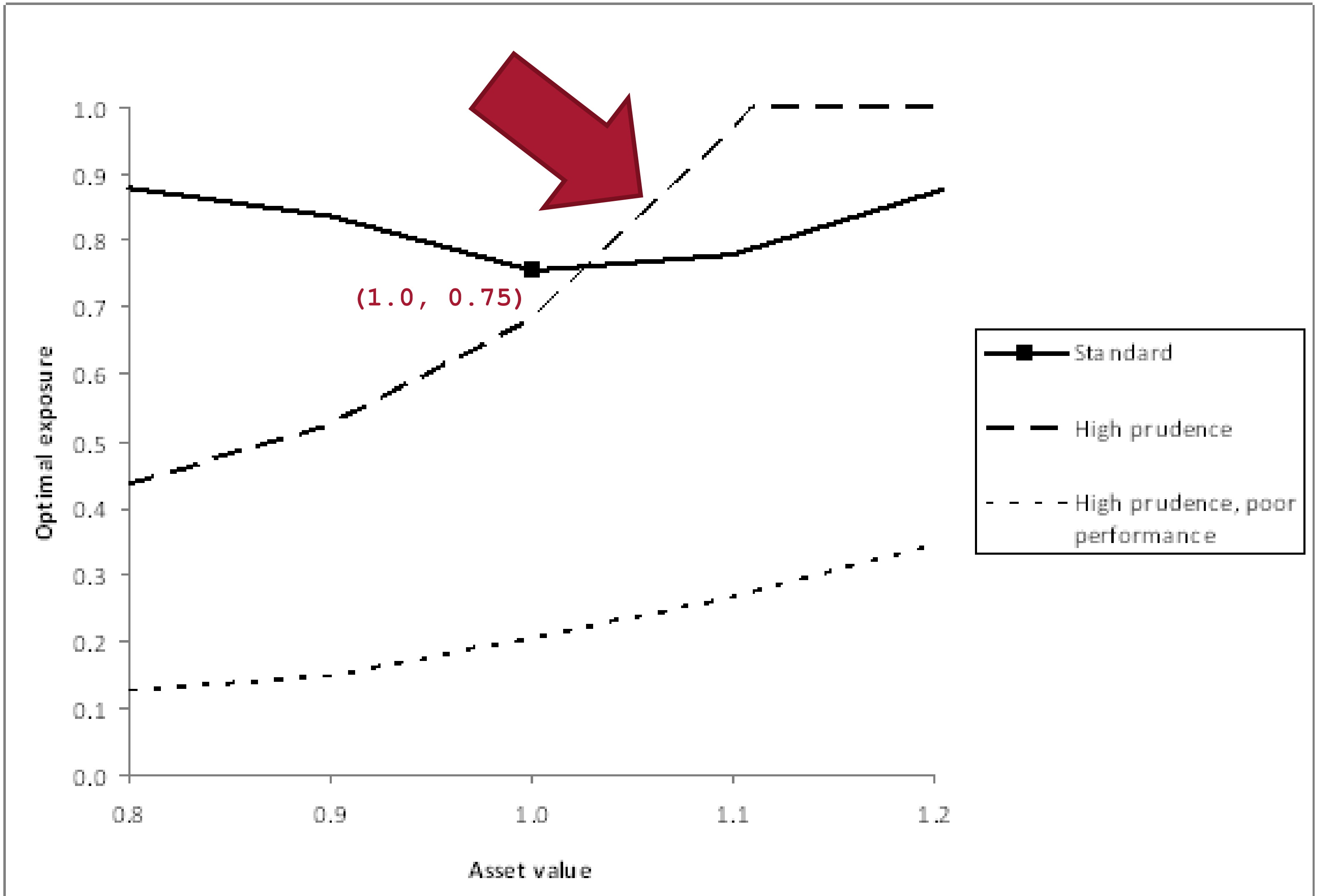

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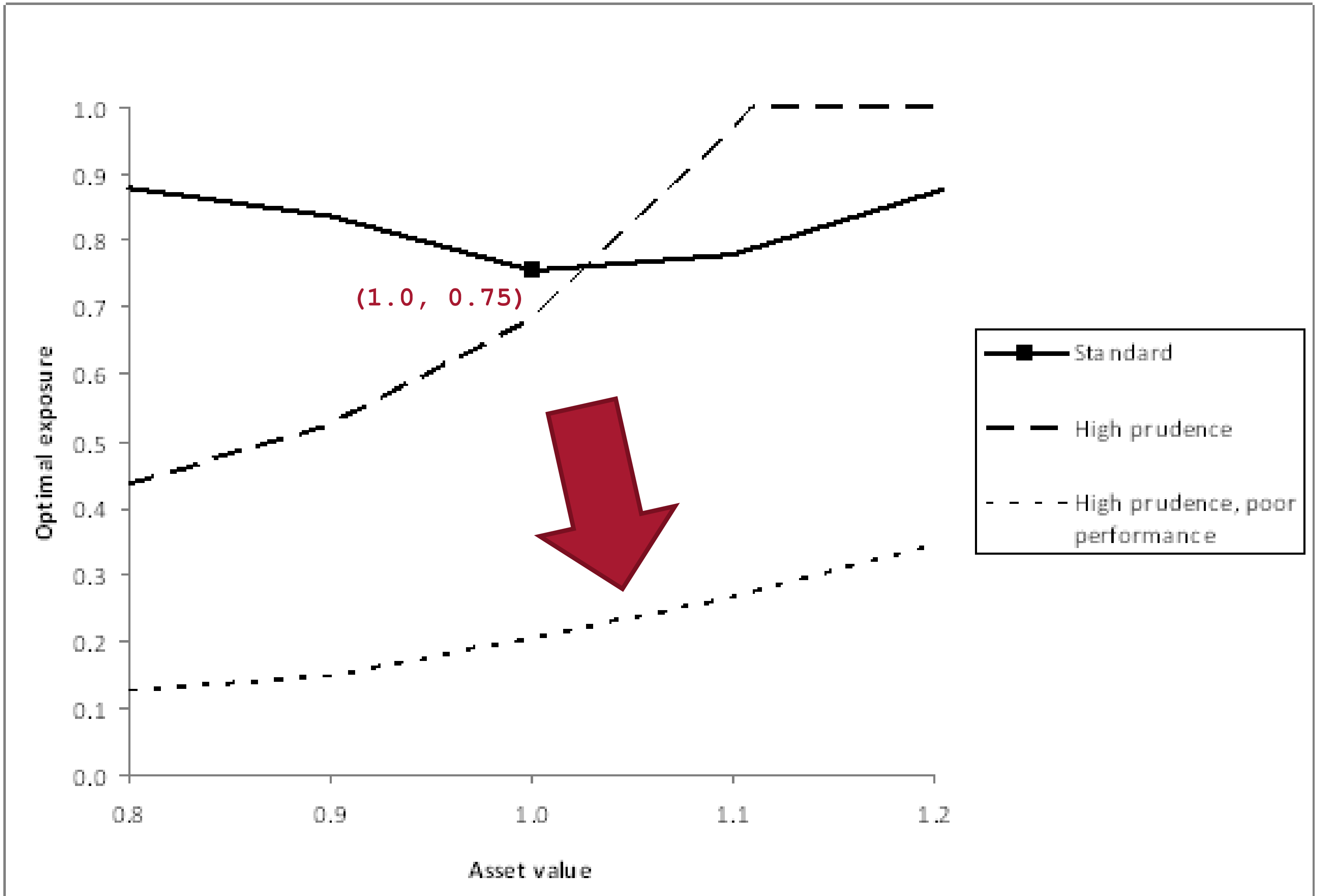

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Summary

Distinction between **type-1 prudence** and **type-2** prudence:

$$E = \int_0^\infty u(z) f(z) dx$$

Summary

Distinction between **type-1 prudence** and **type-2** prudence:

$$E = \int_0^\infty u(z) f(z) dx$$

Summary

Distinction between type-1 prudence and type-2 prudence

DC benefit ratio:

$$z_T = \frac{A_T \cdot {}_{T-t}a_0}{A_0 . a_T}$$

Summary

Distinction between type-1 prudence and type-2 prudence

DC benefit ratio

DB benefit ratio:

$$z_t = \frac{A_t + P_t}{\dot{L}_t + \dot{P}_t}$$

Summary

Distinction between type-1 prudence and type-2 prudence

DC benefit ratio

DB benefit ratio

Criteria required for type-2 prudence:

1) coverage
2) continuity
3) unsatiation
4) relative risk aversion
5) non-increasing relative risk aversion

Summary

Distinction between type-1 prudence and type-2 prudence

DC benefit ratio

DB benefit ratio

Criteria required for type-2 prudence

WARRA-class utility function:

$$u(z) = \frac{u_0(z) + cu_\infty(z)}{1 + c}$$

Summary

Distinction between type-1 prudence and type-2 prudence

DC benefit ratio

DB benefit ratio

Criteria required for type-2 prudence

WARRA-class utility function

Method of fitting a WARRA-class utility function*

* including an algorithm & R code

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Summary

Distinction between type-1 prudence and type-2 prudence

DC benefit ratio

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Criteria required for type-2 prudence

WARRA-class utility function

Method of fitting a WARRA-class utility function*

Method of determining optimal exposure using dynamic asset allocation*

Resolution of counter-intuitive effects previously reported

* including an algorithm & R code

Afterthought

From those in whom the workers' trust is placed,
Prudence is their first demand;
May the stepping-stones to the shore of their dreams
Be laid with prudent hand.

Rob Thomson
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